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Nonlinear mixing of optical vortices with fractional topological charge in Raman sideband generation

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Abstract
We studied the nonlinear parametric interaction of femtosecond fractionally-charged optical vortices in a Raman-active medium. Propagation of such beams was described using the Kirchhoff–Fresnel integrals by embedding a non-integer $2\pi$ phase step in a Gaussian beam profile. When using fractionally-charged pump or Stokes beams, we observed the production of new topological charge and phase discontinuities in the Raman field. These newly generated fractionally-charged Raman vortex beams were found to follow the same orbital angular momentum algebra derived by Strohaber \textit{et al.} (2012 \textit{Opt. Lett.} \textbf{37} 3411) for integer vortex beams.

Keywords: femtosecond, Raman sideband generation, optical vortices

((Some figures may appear in colour only in the online journal)

1. Introduction

It is well-known that light possesses spin angular momentum associated with its polarization state; however, it is less known that it can also possess orbital angular momentum (OAM). Spin angular momentum of light has been studied by Beth [1] in a well-known experiment designed to investigate the transfer of optical angular momentum to matter. Beth’s experiment used birefringent wave plates suspended by a fine quartz wire. A similar experiment using Laguerre–Gaussian (LG) beams was proposed by Allen [2], who suggested the use of a cylindrical lens as a mode converter. Beams of light with OAM such as LG beams are characterized by a central phase singularity, in which an enclosed line integral of the gradient of the phase around the singularity results in an integer $\ell$ number of $2\pi$ phase changes [2, 3]. This is known as the Berry phase and indicates the amount of topological charge (TC) enclosed within the given line integral. In these experiments, the transfer of angular momentum was due to the intrinsic spin or the external degrees of freedom of the field as described by linear optics.

Recently a number of investigations involving the nonlinear interaction of optical vortices with matter have appeared in the literature [4–6]. The first two experiments were carried out by Strohaber \textit{et al.} [4], and Hassinger \textit{et al.} [5]. In [4], two time-delayed chirped pulses embedded with an azimuthal phase from a spiral phase plate were crossed in a PbWO$_4$ crystal, and, in [5], broadband radiation was spectrally separated into two portions, where the difference between the central frequencies of each portion was used in a wave mixing process. In these two original experiments, the authors demonstrated, for the first time, the coherent transfer of optical angular momentum in the generation of cascaded Raman sidebands. The authors of both groups derived an OAM-algebra, or vortex or TC-algebra, which was verified by phase measurements. Later experiments showed that this same algebra [$\ell_{n}^{\text{AS}} = (n + 1)\ell_{P} - n\ell_{S}$ and $\ell_{n}^{\text{S}} = (n + 1)\ell_{S} - n\ell_{P}$] also held when Ince–Gaussian
modes were used [7]. Figure 1 shows an energy level diagram illustrating how pump and Stokes photons mix to generate new photons with new energy and momentum. To produce the first anti-Stokes (AS) order, a pump photon is absorbed leaving the system in a virtual state. The Stokes beam stimulates the emission of a Stokes photon bringing the system to an excited state. This is followed by the absorption of a second pump photon and the spontaneous emission of a new AS photon. For high pumping rates, the newly generated AS radiation participates in pumping the system to higher virtual states, which results in higher frequency AS radiation. Similar arguments can be made for the lower frequency Stokes radiation.

In the present work, we advance upon our earlier work [4, 7] by investigating the production of Raman sidebands with fractionally charged phase singularities. A fundamental difference between previous work, involving optical beam modes with integer charges, and the present work with fractionally charged vortices is that the latter are not self-similar beams and the propagation dynamics are therefore more complex. Fractionally-charged vortices are of interest in laser-beams and the propagation dynamics are therefore more complex. Fractionally-charged vortices are of interest in laser-matter interactions because OAM may influence selection rules. A question that naturally arises is how can a fractional amount of angular momentum be quantum mechanically transferred to matter [8, 9]? As will be shown later, a phase-discontinuity due to a non-integer phase step with a fraction of $2\pi$ produces a beam consisting of an infinite number of integer-charged vortices. In our analysis, imaging of cross sectional distributions of laser beams plays an important role; therefore, we first consider changes in the beam profile with different values of the TC.

2. Propagation of a Gaussian beam with a fractional azimuthal phase

For comparison with the experiment, we theoretically describe the propagation of a Gaussian beam with a fractionally charged azimuthal phase using the Kirchhoff–Fresnel integral. In addition to comparison with the measured data, results obtained in the Fresnel approximation provide an insight into the effects of the phase on the diffracted beam. The theoretical results also elucidate diffraction phenomena typically encountered during the production of optical vortices. For example, it may be surmised that the multi-ringed pattern, often observed in the diffracted beam profile of a Gaussian beam embedded with an azimuthal phase, is due to improper spatial amplitude modulation. One is tempted to expand the distribution in the LG basis as \( \sum_{m=-\infty}^{\infty} c_{m} \psi_{m} \psi_{L}^{\ell} \); however, this superposition does not produce the observed diffraction effects. Our analysis indicates that the multi-ringed diffraction phenomena typically encountered in vortex generation in phase-only production is the result of the phase singularity at the center of the beam. The expression for the electric field, in the Fresnel approximation, of a fractionally charged vortex beam with an embedded azimuthal phase variation of \( \exp(i\theta) \) is (see appendix A),

\[
E(r, z) = \sum_{m=-\infty}^{\infty} c_{m} e^{i\theta} \left\{ e_{0,0} + \frac{\pi}{8} \rho e^{-\frac{1}{4} \rho^2} \right\} \times \left[ I_{a-h/2} \left( \frac{1}{4} \rho^2 \right) - I_{a+h/2} \left( \frac{1}{4} \rho^2 \right) \right].
\]

Here \( c_{m} = (-1)^{m} \sin(\pi a)/[\pi (a - n)] \), \( p^2 = 1/w_{0}^2 + ik/2z \), \( \rho^2 = b^2/2p^2 \), \( b = ik/z \), with \( k \) being the wavenumber, \( z \) the propagation distance, \( w_{0} \) the beam waist, and \( I_{n}(x) \) the modified Bessel function.

Equation (1) is the propagation equation for the electric field amplitude produced by the diffraction of a Gaussian beam through a phase plate having an arbitrary step height. A similar result has been obtained by Berry for the diffraction of a plane electromagnetic wave by a phase plate of arbitrary height [11]. Equation (1) is useful to us as it provides insight into the evolution of fractionally-charged optical vortices. A drawback of using a plane wave is that for small fractional step heights, edge diffraction from the phase mismatch dominates the diffraction pattern. A comparison between a uniform plane wave and a Gaussian beam is shown in figure 2.
for the sequence of TCs of $a = (0.01, 0.1, 0.5, 1.0)$. Equation (1) will be further used to calculate the interferograms of fractionally-charged vortices. Experimental results are expected to show the appearance of lines of lower intensity due to a phase discontinuity, which are ubiquitous features of fractional vortices.

3. Experimental setup

In our experiments, we used amplified femtosecond radiation from a Ti:sapphire chirped-pulse amplification system (Spectra Physics, Spitfire). Our system produces $\sim 50$ fs laser pulses at a repetition rate of 1 kHz. The central wavelength was 800 nm, and the energy per pulse was $\sim 1$ mJ. Output radiation from the laser was sent through a variable iris (not shown) used to control mode quality [7] and beam power. Following the iris, the radiation was sent through the symmetric beam crossing setup shown on the left in figure 3, with the outputs being the time-delayed pump and Stokes beams. In contrast to our previous setups [4, 7], the Michelson interferometer was placed last with one of the mirrors being replaced by a parallel aligned liquid crystal on silicon spatial light modulator (Hamamatsu LCOS-SLM 104683). This configuration allowed us to independently modulate the radiation in both the pump and the Stokes beams [12, 13] and produced two sets of sidebands (solid and dotted lines in figure 3). One arm of the Michelson was used to produce an independent set of Raman sidebands with Gaussian profiles serving as a reference (dotted lines in figure 3) to investigate the phase content of the generated sidebands produced by the other arm (solid lines in figure 3) [4, 5, 7]. This is the simultaneous Young’s double slit experiment [4], and allows for each Raman sideband of different frequency to simultaneously have a reference beam of the same frequency. Figure 3 illustrates this by the solid and dotted beams exiting the PbWO$_4$ crystal. As can be seen in the figure, the two sets of sidebands are being generate at two different but closely spaced locations in the crystal. By displaying a sequence of grayscale images on the SLM, the phase modulation as a function of normalized grayscale value (0 to 1) was found to
have a linear dependence $y = 10.5x$, which for a grayscale value of 0.6 corresponds to a phase shift of $2\pi$ [12]. To produce Raman sidebands, we employed time-delayed pulses in the beam crossing setup (figure 3). Images of the pump and Stokes beams, and the generated Raman orders were captured using an iSight CCD (Apple, iPhone 5s) consisting of 8 megapixels with 1.5 $\mu$m per pixel, and a charge-coupled device (Spiricon, SP503U) having a resolution of 640 $\times$ 480 pixels.

To produce Raman sidebands, the pump and Stokes beams were focused into the Raman-active crystal using a lens having a nominal focal length of 40 cm. The separation distance between the beams at the position of the lens (figure 3) was $\sim 2.25$ cm and resulted in a full-angle of 3.2° between the two beams. The nonlinear crystal was a 0.5 mm thick lead tungstate crystal (MTI Corporation). Lead tungstate (PbWO$_4$) is a uniaxial crystal belonging to the tetragonal crystal group (space group $I4_1/amd$) [14]. Because lead tungstate is a stable, non-hygroscopic and low cost material, it is a popular Raman crystal that has found applications in the construction of Raman lasers [15] and in the generation of sidebands [4, 5]. Lead tungstate has a broad optical transparency spanning the wavelength range from 0.33 to 5.5 $\mu$m [14]. Of particular relevance to the intense femtosecond applications used in this work, PbWO$_4$ has a relatively high damage threshold. Stimulated Raman scattering experiments have been used to obtain spectra over a large wavelength range. The spectrum of PbWO$_4$ is dominated by the totally symmetric (breathing) Ag optical modes of its tetrahedral WO$_4^2$ ions at 900 cm$^{-1}$ [15] and a relatively strong Raman line at 325 cm$^{-1}$ from the internal mode of WO$_4$ [15]. We have previously shown [7] that the frequency difference of the delayed chirped pulses is capable of being tuned to the Raman transitions in a nonlinear medium (figure 1). For linear chirped pulses, the phase content has a quadratic time dependence, so that the instantaneous frequency is given by $\omega = \omega_0 + bt$. Adding a delay $t_d$ between the pulses results in a frequency difference of $\Delta \omega = b t_d$ (figure 1). When this $\Delta \omega$ is equal to the Raman frequency $\Delta \omega_R$ i.e., the difference between the two lowest states in figure 1, we get the production of sidebands. The optimal conditions for sideband generation were found by experimentally scanning the chirp ($b$) and delay ($t_d$) between the two pulses.

4. Experimental results

Using the setup described in the last section, we produced fractionally-charged vortices from a computer generated hologram of an azimuthal phase ramp having an azimuthal phase mismatch. When the phase step was greater than the available phase modulation of the SLM, we used phase wrapping similar to that found in Fresnel lenses. Figure 4(a) shows experimental near field ($z \approx 0.5$ m) images of the intensity and phase of fractionally charged vortices ranging within $a = 0 - 3.4$. We took images in phase steps of 0.1 rad, but present images with phase steps of 0.5 rad. The beams with integer values of $a = \ell$ in columns 1 and 3 are nearly cylindrically symmetric, and exhibit a central multifurcation or region where a fringe branches into multiple fringes. The beams with half-integer values of $a = \ell/2$ in columns 2 and 4 exhibit a line of zero intensity. From the interferograms, it can be seen that the line of zero intensity is associated with a phase discontinuity i.e., where the maxima of the interference fringes abruptly shift by half the period of the interference pattern. These near-field beams were projected onto a white screen and images were taken with an iSight camera. Using equation (1), we simulated the near-field intensity profiles and

**Figure 4.** (a) Experimentally measured near-field interferograms of fractionally-charged vortices produced by an SLM. The fractional charges are indicated in the upper corner of each sub-panel. (b) Theoretical interferograms of near-field fractional vortices calculated using the Kirchhoff–Fresnel integral (equation (1)).
interferograms, figure 4(b). To determine the TCs of the vortices in the far-field at the location immediately before the crystal, we use a 40 cm focal length lens to focus the beams from the SLM. In the focus, we investigated the TC by using the tilted lens method. As with the near-field beams, we used our theoretical calculations (equation (1)) to simulate the intensity profile of the far-field beams shown in rows 1 and 3, figure 5(b). When the TC is an integer value \( a = \ell \), the tilted lens pattern shows a number of nodes equal to the TC. For half-integer vortices, an additional partial lobe appears.

For the integer vortices, we were able to derive exact theoretical expression describing the modal conversion of the vortices to Hermite–Gaussian modes, and for the fractional vortices we found an approximate expression. These solutions are derived in appendix B, and provided us with guidance for finding the optimal experimental configuration. This optimal was achieved by approaching the focal region from downstream with our CCD camera while rotating the lens, so as to improve the separation of the modal lobes in the spatial image and to avoid possible damage to our CCD. The measuring position was inside the focal region, but at a position that resulted in an elliptical shape of the beam as seen in figure 5 for the lowest order Gaussian mode.

To show how the tilted lens generates Hermite Gaussian beams, we use an expression we previously derived in [7] for the modal decomposition of LG beams in the HG basis,

\[
LG_{\ell,k} = N_0^\ell \pi (-1)^{|\ell|} \frac{2}{\pi 2^{|\ell|}} \sum_{k=0}^{|\ell|} \frac{1}{k!(|\ell|-k)!} \times (i \text{sgn}(\ell))^k H_{|\ell|-k,k}.
\]

In equation (2), the LG beam is at the position of the lens \( z = 0 \). For the beam at \( z = z_{\pm}, \) we included only the Gouy phases for each \( HG_{|\ell|-k,k} \) in the sum

\[
\Psi_{0,\ell}^{\text{far-field}} = N_0^\ell \pi (-1)^{|\ell|} e^{i(\ell\Phi + \Phi_0)} \frac{2}{\pi 2^{|\ell|}} \sum_{k=0}^{|\ell|} \frac{1}{k!(|\ell|-k)!}(i \text{sgn}(\ell))^k \times H_{|\ell|-k,k} e^{-ik(\ell\Phi + \Phi_0)}.
\]

Equation (3), gives the field distributions for the HG beams in figure 5(b), rows (2) and (4). For the non-integer optical vortices, the analytical solution is somewhat difficult to find, but we made an approximation to zero order

\[
F = \sum_{\ell=-\infty}^{\infty} \epsilon_\ell \Psi_{0,\ell}^{\text{far-field}}.
\]

Here the \( \epsilon_\ell \) are those given in equation (1). The results of equation (4) are plotted in figure 5 and show good qualitative agreement with the experimental results.

Before investigating the production of Raman sidebands with fractionally-charged optical vortices, we produced a series of integer vortices in the pump and Stokes beams to test our new setup. Interferograms of generated sidebands in a simultaneous Young’s double slit experiment definitively verified that the sidebands followed the expected OAM-algebra derived by us [4]. In figure 6(a), the pump beam was embedded with \( \ell_p = 0 \) units (\( b \)) of OAM and the Stokes beam with \( \ell_s = 1 \). Based on the OAM-algebra \( \ell_{\text{ASN}} = (N+1)\ell_{\text{P}} - N\ell_{\text{S}} \), the AS orders are expected to possess TC of \( \ell_{\text{ASN}} = -N\ell_s \). The interferograms in column 2 of figure 6(a) show agreement with the expected OAM content. This agreement can be seen by counting the number of bifurcations in the recorded interferograms. To aid the reader, we have placed red dots at the location of the bifurcations. For example, in the AS3 order in figure 6(a), the expected magnitude of the TC is 3 for which we count three bifurcations. The data in
Figure 6. (a) Images of Raman sidebands in the first three anti-Stokes orders. The pump was encoded with $\ell_p = 0$ units of OAM and the Stokes with $\ell_S = 1$. The second column in panel (a) are interferograms verifying the phase content of the beams. (b) Same as (a) except with $\ell_p = 1$ and $\ell_S = 0$. The location of the bifurcations is indicated by the red dots.

Figure 7. Images of Raman sidebands in the first two anti-Stokes orders. The pump was encoded with $\ell_p = -1$ units of OAM and the Stokes with $\ell_S = 1$. The interferograms verify the OAM algebra $\ell_{\text{ASN}} = (N + 1)\ell_p - N\ell_S$.

Figure 8. (a) Images of Raman sidebands in the first three anti-Stokes orders. The pump was encoded with $\ell_p = 0$ units of OAM and the Stokes with $\ell_S = 1$. The second column in panel (a) are interferograms verifying the phase content of the beams. (b) Same as (a) except with $\ell_p = 1$ and $\ell_S = 0$. The location of the bifurcations is indicated by the red dots.

In our first experiments producing sidebands with fractional vortices, we used a pump beam embedded with a fractional charge of $\ell_p = 3/4$ and a Stokes beam with $\ell_S = 1$. From the OAM-algebra, we expect to get $\ell_{\text{AS1}} = 2\ell_p - \ell_S = 1/2$ in the first AS order. Panels (a) and (b) in figure 8 show the intensity profile and interferogram in this order. Visual inspection of the interference fringes in figure 8(b) indicates that the AS1 order has a TC of $\ell = 0.5$ as seen by the line phase dislocation. For comparison, the intensity profile and interferogram of an $\ell = 0.5$ vortex is shown in figures 8(e) and (f).

Since this is the first reported production of a fractionally-charged vortex in Raman sideband generation, definite verification is required. Lineouts (figure 8(b)) were taken on both sides of the phase discontinuity (dashed lines in figure 8(b)), and a running correlation was calculated figure 8(d). This correlation shows that the fringes to the left of the beam in figure 9(b) are in phase with each other (+1), while fringes to the right are out of phase (−1). This result is expected from a beam with $\ell = 0.5$ units of OAM.
Figure 8. Panels (a) and (b): Intensity profile and interferogram in the first anti-Stokes order. The pump and Stokes beams contained $\ell_p = 3/4$ and $\ell_S = 1$ units of OAM respectively, and the OAM algebra predicts an OAM content of $\ell_{CS} = 1/2$. (c) Lineouts of fringes on either side of the phase dislocation. (d) Running correlation of the line-outs shown in (c). The correlation indicates that the fringes begin in phase $(+1)$ in the first half of the image and are out of phase $(-1)$ in the second half. Panels (e)–(h) are simulated of the experimental data shown in panels (a)–(d).

Figure 9. Production of new topological charge from phase discontinuities. The value in the lower left corner of the images is the topological charge of the Stokes beam. For values of $\ell_S$ near 0.5 and 1.5 a phase discontinuity can be observed followed by the appearance of a bifurcation. Row (5), columns 2 and 3 are an observed phase discontinuity circled in the panels of row 1, column 5 and row 4, column 2. The last two images were obtained using the tilted lens method.
Panels (e)–(h) are simulations of those shown in panels (a)–(d).

As the simplest example, we have observed and verified the generation of a fractionally-charged Raman vortex with TC of 1/2. Next we demonstrate how new TC is produced in the sidebands from phase discontinuities similar to those shown in figure 4. To do this, we produced higher order fractionally-charged vortices by embedding the pump radiation with \( \ell_p = 1 \) and varying the OAM of the Stokes radiation from \( \ell_S = 0 \) to \( \ell_S = -2 \) in steps of 0.1 radian. From the OAM-algebra, we expect the radiation in the AS1 order to possess TCs ranging from \( \ell_{AS1} = 2 \) to \( \ell_{AS1} = 4 \). Figure 9 shows the experimental interferograms for this sequence of pump and Stokes beams. Our first observation is that for integer values of \( \ell_S \) no multifurcations appear, which can be seen in rows 1–5, columns 1–5; and row 5, column 1. When the Stokes beam has an OAM content of around \( \ell_S = 1.6 \) (row 4, column 2, and row 5, column 3), we have traced the formation of new TC from other combinations of pump and Stokes beams, and for which similar results were obtained. Two observations were made: first, the multifurcations in the Raman sidebands were broken-up into single bifurcations, and second, a ‘short’ phase dislocation was usually observed before the appearance of a new bifurcation. We speculate that the absence of a ‘long’ phase discontinuity in the Raman sidebands (as seen in figure 4) with higher order OAM is due to coherent background instabilities [18], which spatially splits the OAM charge into separate bifurcations.

**5. Discussion and summary**

In conclusion, we have for the first time produced fractionally-charged vortices in Raman sidebands. The effect of diffraction on fractionally-charged vortices was investigated, and the TCs were determined by the simultaneous Young’s double slit experiment and the tilted lens methods. The observed fractional charges in the sidebands were found to follow the same OAM algebra that was earlier established for helical-LG with integer charges. The field of future applications for fractionally-charged vortex beams is wide open, possibly spanning novel strong field physics to new pathways in optical excitations.

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**Appendix A**

An analytical solution using the Kirchhoff-Fresnel diffraction integral is found by embedding an azimuthal phase variation \( \exp(i\alpha \theta) \) that is in general not an integer multiple of \( 2\pi \), but an arbitrary real number, into a Gaussian beam amplitude

\[
\psi(x, y, 0) = N_0^0 e^{-r^2/w_0^2} e^{i\alpha \theta}.
\]

Here \( N_0^0 = \sqrt{\pi/2w_0^2} \) normalizes the intensity of the transverse beam, \( w_0 \) is the beam waist and \( a \) is any real number. This beam can be propagated using the Fresnel integral in cylindrical coordinates,

\[
E(r, z) = -\frac{i}{\lambda z} e^{ikr} \exp\left(-\frac{ikr^2}{2z}\right) \int_0^{\infty} \int_0^{2\pi} d\theta' r' dr' \\
\times \left(N_0^0 e^{-r'^2/w_0^2} e^{i\alpha \theta'}\right) \exp\left(-\frac{1}{2} \frac{kr'^2}{z} \cos(\theta - \theta')\right).
\]

The angular integral can be evaluated by first making a Fourier series expansion of the fractional phase factor,

\[
e^{i\alpha \theta} = \sum_{n=-\infty}^{\infty} c_n e^{in\theta} = \sin(\pi a) \frac{1}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \frac{1}{a - n} e^{i\alpha \theta},
\]

where \( n \) is any integer number. To compress the equations and make them more transparent, we make the following substitutions: \( c_m = (-1)^m \sin(\pi a)/[\pi(a - n)] \), \( p^2 = 1/w_0^2 + ik/2z \), and \( b = ikr/z \). With these substitutions and the Fourier expansion in equation (A.3), the integral in equation (A.2) reduces to,

\[
E(r, z) = A \sum_{n=-\infty}^{\infty} c_n e^{i\alpha \theta} \int_0^{\infty} \int_0^{2\pi} d\eta' r' dr' e^{-p^2 r'^2}.
\]

Here the angular variable has changed since we have made the substitution \( \eta = \theta - \theta' \). The complex exponential having a cosine argument can be expanded using the well-known Jacobi–Anger expansion,

\[
e^{ibr' \cos(\eta)} = \sum_{m=-\infty}^{\infty} i^m L_m(br') e^{im\eta}.
\]
Using equation (A.5), the angular integral can be performed
\[ E(r, z) = 2\pi A \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} i^n c_n e^{i\alpha_n} \int_0^{\infty} r'dr'J_m(b r') e^{-\rho^2 r'^2} \] (A.6)

In order to carry out the integration over the radial spatial coordinate, we integrate by parts one time,
\[ \int_0^{\infty} r'dr'J_m(b r') e^{-\rho^2 r'^2} = -\frac{1}{2\rho^2} J_m(b r') e^{-\rho^2 r^2} \bigg|_0^{\infty} + \frac{1}{2\rho^2} \times \int_0^{\infty} dr' e^{-\rho^2 r'^2} \frac{d}{dr'} J_m(b r'). \] (A.7)
The first term on the right is non-zero when \( m = 0 \) otherwise it is equal to \( \delta_{m,0}/2p^2 \) after evaluating the integration bounds.

Using the recurrence relation \( 2dU_n(x)/dx = J_{n-1}(x) - J_{n+1}(x) \), the last integral reduces to
\[ \int_0^{\infty} r'dr'J_m(b r') e^{-\rho^2 r'^2} = \frac{\delta_{m,0}}{2p^2} + b \frac{4p^2}{4p^2} \int_0^{\infty} dr' e^{-\rho^2 r'^2} \left[ J_{m-1}(b r') - J_{m+1}(b r') \right]. \] (A.8)
The remaining integral over the radial coordinate can be carried out using a modified version of Hankel's more general integral [10]
\[ \int_0^{\infty} J_m(b r') e^{-\rho^2 r'^2} dr' \approx \frac{\sqrt{\pi}}{2p} \exp \left( -\frac{b^2}{8p^2} \right) I_{m/2} \left( \frac{b^2}{8p^2} \right) \] (A.9)

Using equation (A.9) we arrive at
\[ \int_0^{\infty} r'dr'J_m(b r') e^{-\rho^2 r'^2} \approx \frac{\delta_{m,0}}{2p^2} + \frac{\rho}{2p^2} \frac{\sqrt{\pi}}{8e} \frac{1}{4\rho^2} \left[ I_{m-1/2} \left( \frac{1}{4\rho^2} \right) \right] - I_{m+1/2} \left( \frac{1}{4\rho^2} \right). \] (A.10)

Here \( I_n(x) \) are the modified Bessel functions and \( \rho^2 = b^2/2p^2 \). By combining equations (A.6) and (A.10), the final expression for the electric field in the Fresnel approximation for a fractionally charged vortex is
\[ E(r, z) = \pi A \frac{1}{p} \sum_{n=-\infty}^{\infty} \left\{ \delta_{n,0} + \frac{\sqrt{\pi}}{8} \rho e^{-1/4\rho^2} \right\} \left[ I_{n-1/2} \left( \frac{1}{4\rho^2} \right) - I_{n+1/2} \left( \frac{1}{4\rho^2} \right) \right]. \] (A.11)

Notice from equation (3) that when \( n = 0 \) the expansion coefficient is zero \( c_0 = 0 \), and because \( \delta_{n,0} = 0 \) when \( n \neq 0 \), we can neglect the first term \( \delta_{n,0} \) in the bracketed expression.

**Appendix B**

When a lens is tilted, the focal length of the beam changes along the axis about which the lens is being rotated and along the axis perpendicular to the rotation axis and the beam [16]. If the lens is rotated about the y-axis, the diameter of the lens along that axis is unchanged \( y' = y \), but the diameter of the lens in the x-direction effectively shrinks as \( x' = x \cos \theta \), where \( \theta \) is the tilt angle. In both cases, the thickness of the lens at its ‘center’ increases as \( d'/\cos(\theta) \). From these relationships and using the Lensmaker’s equation for a plano-convex lens, we found, in the approximation that \( d/R \ll 1 \), the two focal lengths
\[ f_x \approx f_{\text{len}} \cos^2(\theta), \quad f_y \approx f_{\text{len}} \cos(\theta). \] (B.1)

Plots of the effective focal lengths in the \( x \) and \( y \) directions are shown in figure B.1(a). In our experiments, we use a lens with a focal length of 40 cm. This graph was used to guide us in adjusting the tilt angle of the lens. To determine the required focal length difference, we used Gaussian optics.

When a Gaussian beam is focused, the effective focal length \( f = f(1 + f^2/z_0^2) \) and the waist \( w_0 = w_{0L}/\sqrt{2} + f^2 \) are given in terms of the focal length \( f \) of the lens, \( f \) and the waist \( w_0 \) and Rayleigh range \( z_0 \) of the input beam [17]. Upon tilting the lens, the focal lengths along the \( x \) direction \( f_x \) and along the \( y \) direction \( f_y \) are not equal, therefore the beam parameters are not equal. The Rayleigh ranges \( z_{0x} \) and \( z_{0y} \) of the beam in the \( x \) and \( y \) directions are found by setting the beam spot sizes \( w_x = w_{0L}/\sqrt{1 + (z - l_x)^2/z_{0x}^2} \) and \( w_y = w_{0L}/\sqrt{1 + (z - l_y)^2/z_{0y}^2} \) equal to \( w_{01} \) at the position of the lens \( z = 0 \). In this way, the Rayleigh ranges in the \( x \) and \( y \) directions are found to be \( z_{0x} = l_x/\sqrt{w_{0L}^2/w_{0y}^2 - 1} \) and \( z_{0y} = l_y/\sqrt{w_{0L}^2/w_{0x}^2 - 1} \). We wish to find the position in the focus where \( w_x = w_{01} \) and the Gouy phase difference is equal to \( \Phi_G - \Phi_G = \pi/2 \). This can be achieved by setting the derivative of the Gouy phase difference equal to zero
\[ \frac{d}{dz} \left[ \arctan \left( \frac{z - l_x}{z_{0x}} \right) - \arctan \left( \frac{z - l_y}{z_{0y}} \right) \right] = 0. \] (B.2)

After some algebra, the \( z \) position of the maximum of the Gouy phase difference can be expressed by either of the following two solutions,
\[ z_+ = l_x + z_{0y} \frac{l_x - l_y}{z_{0x} - z_{0y}} \left[ -1 + \frac{z_{0y}}{z_{0x}} \right] \left( 1 + \frac{(z_{0x} - z_{0y})^2}{(l_x - l_y)^2} \right) \]
\[ z_- = l_y - z_{0x} \frac{l_x - l_y}{z_{0y} - z_{0x}} \left[ -1 + \frac{z_{0x}}{z_{0y}} \right] \left( 1 + \frac{(z_{0y} - z_{0x})^2}{(l_y - l_x)^2} \right). \] (B.3)

By setting the operand in equation (B.2) equal to \( \pi/2 \) and by taking the \( z \) positions to be those of equation (B.3), the difference in effective focal lengths between the \( x \) and \( y \)
directions is given by the simple expression

$$l_x - l_y = 2\sqrt{f_x f_y}.$$  \hspace{1cm} (B.4)

In figure B.1(b) we have plotted the beam spot sizes in the $x$ (solid blue curve) and $y$ (solid red curve) directions, the associated Gouy phases and the Gouy phase difference (solid black curve). Using equation (B.4), we find that the difference in the focal lengths for a $\pi/2$ phase difference is $f_x - f_y \approx 10$ cm. From figure B.1(a) this corresponds to a lens tilt angle of about $\sim 35^\circ$.

References