Mode-locking

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Overview

• Basic Idea of mode-locking
• How modelocking is achieved
  – Active
  – Passive
• Theory
• Examples of real lasers
• Dispersion
• Pulse measurement
Mode-locking – Basic Idea
Mode-locking – Basic Idea

\[ I = (E_A + E_B + E_C)^2 \]

\[ \frac{2L}{c} \]
Active Mode-locking

LOCKING OF He-Ne LASER MODES INDUCED BY SYNCHRONOUS INTRACAVITY MODULATION

(diffraction by phonons in crystals; E)

L. E. Hargrove, R. L. Fork, and M. A. Pollack
Bell Telephone Laboratories, Incorporated
Murray Hill, New Jersey
(Received 11 May 1964)

Fig. 1. Schematic diagram of the experimental arrangement.

Fig. 2. A typical locked optical mode spectrum.
Variations of Active Mode-locking

- Synchronous phase modulation
- Regenerative
- Harmonic
- Synchronously pumped laser
- Hybrid
Passive Mode-locking using saturable absorbers

- An intra-cavity saturable absorber is able to fix phase of cavity modes

- Variations
  - Slow Saturable Absorber
  - Fast Saturable Absorber
    - Kerr-lens effect
Figure 2.9 Colliding-pulse mode-locked ring dye laser. OC, output coupler.
Slow Saturable Absorber

- Sub-picosecond pulse generation with passive mode-locking first demonstrated in early 1970s
  - Dye laser
- <100 femtosecond pulses first demonstrated with a colliding pulse mode-locked ring dye laser
- Relaxation times for dyes typically on order of nanoseconds
  - How pico/femtosecond pulses can be produced from such long absorber lifetime?
  - Theory of Haus in mid-1970s

![Image](image_url)  
Figure 2.7 Pulse-shortening process in slow saturable absorber mode-locking. The shaded region indicates net positive gain.
Active Modelocking

Figure 2.1  (a) Actively mode-locked laser arrangement, with an intracavity modulator driven at the cavity round-trip period; (b) periodic modulator transmission and resulting mode-locked pulses.

Figure 2.2  Model of an actively mode-locked laser.
Passive Modelocking

• The action of passive modelocking can be modeled using a time-dependent gain and loss
  – Modelocked pulse is self-consistent solution

**Figure 2.5** Model of a passively mode-locked laser.
Suppose we have an initial pulse envelope $a(t)$ with Fourier transform $A(\omega)$. 

Quantitative Analysis – Passive Modelocking Equation
Modelocking Equation

• We take into account cavity by writing

\[ A'(\omega) = e^{-l_0} e^{-\tilde{\omega}^2/\omega_c^2} A(\tilde{\omega}) \]

  - Here \( l_0 \) is the linear time independent loss
  - Filtering term \( e^{-\tilde{\omega}^2/\omega_c^2} \) describes the finite bandwidth of the cavity, approximated as gaussian

• Assuming changes by each element is small, and the modelocked spectrum is small compared to \( \omega_c \)

\[ A'(\tilde{\omega}) \approx \left( 1 - l_0 - \frac{\tilde{\omega}^2}{\omega_c^2} \right) A(\omega) \]
Modelocking Equation

\[ A'(\tilde{\omega}) \approx \left( 1 - l_0 - \frac{\tilde{\omega}^2}{\omega_c^2} \right) A(\omega) \]

- Taking the Fourier transform of this expression

\[ a'(t) = \left( 1 - l_0 + \frac{1}{\omega_c^2} \frac{d^2}{dt^2} \right) a(t) \]

- We now introduce nonlinear loss and gain
  - Time dependent gain \( g(t) \)
  - Time dependent loss \( l(t) \)

- Nonlinear loss and gain are small per pass

\[ a''(t) = e^{-l(t)} e^{g(t)} a'(t) \approx a \left( 1 - l_0 - l(t) + g(t) + \frac{1}{\omega_c^2} \frac{d^2}{dt^2} \right) a(t) \]
Modelocking Equation

- We now require that the mode-locked pulse be reproduced after cavity round trip.

- $a''(t)$ equal to $a(t)$, except for temporal shift $\delta T$

$$a''(t) = a(t + \delta T) \approx a(t) + \frac{da}{dt} \delta T$$

- We then arrive at

$$\frac{1}{\omega_c} \frac{d^2 a(t)}{dt^2} + [g(t) - l(t) - l_0] a(t) - \delta T \frac{da(t)}{dt} = 0$$

- This is the Haus master equation for modelocking.
Saturable Absorber Model

Figure 2.6  Model of a four-level saturable absorber.
Saturable Absorber Model

Figure 2.6 Model of a four-level saturable absorber.
• Rate equation for level 1

\[ \frac{\partial N_1}{\partial t} = \frac{N_3}{\tau_A} - \frac{\sigma_A |a(t)|^2}{\hbar \omega_0 A_A} (N_1 - N_2) \]

• With the approximation of fast transitions, we make the assumption

\[ N_1 + N_2 = N_A \quad N_2 \approx N_4 \approx 0 \]

\[ \frac{\partial N_1}{\partial t} = \frac{N_A - N_1}{\tau_A} - \frac{|a(t)|^2 N_1}{P_A \tau_A} \]

\[ P_A = \frac{\hbar \omega_0 A_A}{\sigma_A \tau_A} \]
Modelocking Equation

• We can then see that the time-dependent loss is proportional to ground-state absorber density

\[ l(t) = \frac{\sigma_A}{2} N_1(t) l_a \]

• We can then analyze the dynamics of \( N_1(t) \) in the limiting cases of fast and slow saturable absorbers
Figure 2.7  Pulse-shortening process in slow saturable absorber mode-locking. The shaded region indicates net positive gain.
Slow Saturable Absorber

\[
\frac{\partial N_1}{\partial t} = \frac{N_A - N_1}{\tau_A} - \frac{|a(t)|^2 N_1}{P_A \tau_A}
\]

- For a slow saturable absorber \( \tau_A \) is large, therefore we can take

\[
\frac{N_A - N_1}{\tau_A} \approx 0
\]

- We then obtain

\[
\frac{\partial N_1}{\partial t} = -\frac{|a(t)|^2 N_1}{P_A \tau_A}
\]

Figure 2.6 Model of a four-level saturable absorber.
Slow Saturable Absorber

\[
\frac{\partial N_1}{\partial t} = -\frac{|a(t)|^2 N_1}{P_A \tau_A}
\]

- With the use of an integrating factor, this is easily solved:

\[
N_1(t) = N_1^{(i)} e^{-\int^t dt |a(t)|^2 / P_A \tau_A} = N_1^{(i)} e^{-U(t)/U_A}
\]

- Here we have the pulse energy at time t, and the saturation energy

\[
U(t) = \int^t dt |a(t)|^2 \quad \quad U_A = P_A \tau_A
\]

Note that in the slow absorber case, it is the saturation energy, not saturation intensity that is important.
Slow Saturable Absorber

- After the pulse interacts with the absorber, the population relaxes exponentially back to equilibrium

\[ N_1(t) = N_A + \left( N_1^{(i)} e^{-U/U_A} - N_A \right) e^{-t/T_A} \]

- We can then write the time dependent loss as

\[ l(t) = l^{(i)} e^{-U(t)/U_A} \approx l^{(i)} \left[ 1 - \frac{U(t)}{U_A} + \frac{1}{2} \frac{U^2(t)}{U_A^2} \right] \]

Here we have assumed that the pulse energy is small enough to allow for a second order Taylor expansion
Slow Saturable Absorber

\[ l(t) = l^{(i)} e^{-U(t)/U_A} \approx l^{(i)} \left[ 1 - \frac{U(t)}{U_A} + \frac{1}{2} \frac{U^2(t)}{U_A^2} \right] \]

- Following the pulse, the loss recovers exponentially to small signal value

\[ l^{(i)} = L_{\text{sat}}^{(0)} + \left( l^{(i)} e^{-U/U_A} - l_{\text{sat}} \right) e^{-T/\tau_A} \]

\[ U = \text{Total Pulse Energy} \]
\[ T = \text{Cavity Round Trip Time} \]

Figure 2.7  Pulse-shortening process in slow saturable absorber mode-locking. The shaded region indicates net positive gain.
Time Dependent Gain

- We analyze gain for a four level laser system

Figure 1.5  Energy-level structure for a four-level atom.
Time Dependent Gain

- With the previous assumptions, the rate equation for the gain medium is

\[ \frac{\partial N_3}{\partial t} = W(N_G - N_3) - \frac{N_3}{\tau_G} - \frac{|a(t)|^2}{P_G \tau_G} N_3 \]

- We only need to concern ourselves with slow saturable gain media \( t_p \ll \tau_G \)

\[ g(t) = g^{(i)} e^{-U(t)/U_G} \]

Gain just before pulse

- After this occurs, gain recovers exponentially

\[ g(t) = \left( g^{(i)} e^{-U/U_G} - g_0 \right) e^{-t/\tau_G} + g_0 \]

Small signal gain
Time Dependent Gain

- We analyze gain for a four level laser system

![Energy-level structure for a four-level atom](image)

**Figure 1.5** Energy-level structure for a four-level atom.
Slow Saturable Absorber – Net Gain

- The time-dependent net gain is then

\[ g_T(t) = g(t) - l(t) - l_0 \]

\[ l(t) = l^{(i)} e^{-U(t)/U_A} \approx l^{(i)} \left[ 1 - \frac{U(t)}{U_A} + \frac{1}{2} \frac{U^2(t)}{U_A^2} \right] \]

\[ g(t) = g^{(i)} e^{-U(t)/U_G} \]

\[ l^{(i)} = L^{(0)}_{\text{sat}} + \left( l^{(i)} e^{-U/U_A} - l_{\text{sat}} \right) e^{-T/\tau_A} \]

\[ g(t) = \left( g^{(i)} e^{-U/U_G} - g_0 \right) e^{-t/\tau_g} + g_0 \]

**Figure 2.7** Pulse-shortening process in slow saturable absorber mode-locking. The shaded region indicates net positive gain.
Slow Saturable Absorber – Observations

• Net gain must be negative before and after pulse for stability. Therefore
\[ g^{(i)} < l_0 + l^{(i)} \approx l_0 + l_{\text{sat}}^{(0)} \]

• And
\[ g^{(i)} e^{-U/U_G} < l_0 + l_{\text{sat}}^{(0)} e^{-U/U_A} \]

• For self-starting, we must have
\[ g_0 > l_0 + l_{\text{sat}}^{(0)} \]
Slow Saturable Absorber – Observations

- Net gain must be negative before and after pulse for stability. Therefore

\[ g^{(i)} < l_0 + l^{(i)} \approx l_0 + l^{(0)}_{\text{sat}} \]

- And

\[ g^{(i)} e^{-U/U_G} < l_0 + l^{(0)}_{\text{sat}} e^{-U/U_A} \]

- For self-starting, we must have

\[ g_0 > l_0 + l^{(0)}_{\text{sat}} \]

Gain cannot recover completely between pulses!

\[ g^{(i)} < g_0 \]
To achieve a net gain window the absorber must saturate before the gain:

\[
\frac{g^{(i)}}{U_G} < \frac{l^{(0)}_{\text{sat}}}{U_A}
\]

This can be achieved by focusing more tightly on the absorber.
Slow Saturable Absorber – Analytic Treatment

- Mode locking equation becomes

\[ \frac{1}{\omega_c^2} \frac{d^2 a(t)}{dt^2} - \delta T \frac{da(t)}{dt} + g_T(t) a(t) = 0 \]

- Where

\[ g_T(t) = g(t) - l(t) - l_0 \approx g^{(z)} \left( 1 - \frac{U(t)}{U_G} \right) - l_{\text{sat}}^{(0)} \left( 1 - \frac{U(t)}{U_A} + \frac{1}{2} \frac{U^2(t)}{U_A^2} \right) \]

**Figure 2.8** Time-dependent loss $\ell(t)$, gain $g(t)$, and net gain $g_T(t)$ plotted as a function of pulse energy $U(t)$. The shaded regions indicate net positive gain.
Slow Saturable Absorber – Analytic Treatment

• Symmetric solution of modelocking equation

\[
\frac{1}{\omega_c^2} \frac{d^2 a(t)}{dt^2} - \delta T \frac{d a(t)}{dt} + g_T(t) a(t) = 0
\]

• Is given by hyperbolic secant function

\[
a(t) = \frac{a_0}{\cosh(t/t_P)} = a_0 \text{sech} \left( \frac{t}{t_P} \right)
\]

• Time dependent pulse energy is then

\[
U(t) = \int^t dt = \frac{U}{2} \left[ 1 + \tanh \left( \frac{t}{t_P} \right) \right]
\]
Slow Saturable Absorber Modelocked Laser

Figure 2.9  Colliding-pulse mode-locked ring dye laser. OC, output coupler.
Fast Saturable Absorber

- Difficult to find materials with relaxation times faster than femtosecond pulses!
- Optical Kerr effect can simulate a fast absorber
Fast Saturable Absorber

- We begin again with the mode-locking equation

\[ \frac{1}{\omega_c} \frac{d^2 a(t)}{dt^2} + \left[ g(t) - l(t) - l_0 \right] a(t) - \delta T \frac{da(t)}{dt} = 0 \]

- For solid state lasers we have
  - Low gain cross sections
  - Long relaxation times, \( \tau_G \) typically on order of microseonds to milliseconds
    - Dynamic gain saturation small, replace \( g(t) \) by constant value \( g \)
    - \( g \) is function of small-signal gain \( g_0 \)
Fast Saturable Absorber

- Time Dependent loss becomes

\[ l(t) = \frac{l^{(i)}}{1 + |a(t)|^2 / P_A} \approx l^{(i)} \left[ 1 - \frac{|a(t)|^2}{P_A} \right] \]

- Resulting modelocking equation becomes

\[
\left[ \frac{1}{\omega_c} \frac{d^2}{dt^2} - \delta T \frac{d}{dt} + \left( g - l_0 - l^{(i)} \right) + l^{(i)} \frac{|a(t)|^2}{P_A} \right] a(t) = 0
\]

- Symmetric solution is again the hyperbolic secant:

\[ a(t) = a_0 \text{sech} \left( \frac{t}{t_P} \right) \]
Fast Saturable Absorber

- Substituting solution back into modelocking equation, we get

\[
\frac{1}{\omega_c^2 t_p^2} + g - l_0 - l^{(i)} = 0
\]

Negative gain before and after pulse

\[
-\frac{2}{\omega_c t_p^2} + \frac{l^{(i)} a_0^2}{P_A} = 0
\]

Peak modelocked power inversely proportional to pulse width squared

\[
\delta T = 0
\]

No time shift from SAM

**Figure 2.10** Gain and loss dynamics in fast saturable absorber mode-locking. The shaded region corresponds to positive net gain.
Pulse width and Power

- Substituting

\[- \frac{2}{\omega_c t_p^2} + \frac{l^{(i)} a_0^2}{P_A} = 0 \Rightarrow \frac{1}{\omega_c^2 t_p^2} + g - l_0 - l^{(i)} = 0\]

- We obtain

\[l_0 + l^{(i)} - g = \frac{l^{(i)} a_0^2}{2} \frac{1}{P_A}\]

- Using gain saturation equation

\[g = \frac{g_0}{1 + \langle P \rangle / P_G}\]
We obtain

\[ l_0 + l^{(i)} - \left[ 1 + \left( \frac{2P_A}{l^{(i)}} \right)^{1/2} \frac{2a_0}{\omega_c P_G T} \right]^{-1} g_0 = \frac{l^{(i)} a_0^2}{2P_A} \]

**Figure 2.11** Graphical solution of eq. (2.69), in which the left- and right-hand sides of the equation are plotted vs. pulse amplitude \( a_0 \). Two cases are plotted: the case of a relatively large \( g_0 \) where no single-pulse solutions are found, and the case of a smaller \( g_0 \) where two solutions are found (marked by circles). Of these, only the longer-pulse, lower-peak-power solution is stable.
Figure 2.14  (a) Cavity configuration for the original self mode-locked Ti:S laser (the inset shows the intracavity prism sequence for dispersion compensation); (b) sub-10-fs Ti:S laser in which the prism pair is replaced by chirped, dispersion-compensating mirrors. G, gain medium (Ti:S crystal); OC, output coupler; F, birefringent filter. (a) Adapted from [52]; (b) adapted from Fig. 5 of [55], with permission from Springer Science and Business Media.
5.9 Femtosecond Ti:Sapphire Laser

Fig 1. Left: Diagram of the prismless short ring laser cavity operating at 2.12 GHz. The two mirrors adjacent to the 3 mm-long crystal are curved (3 cm ROC) and the length between the other flat mirrors is 3 cm. Right: Laser output power as function of the pump power at 532 nm.

Obtained after pulse compression
Compression Using MIIPS

- Multiphoton Intrapulse Interference Phase Scan

Fig. 3. MIIPS traces of the first (a) and last (b) iterations. Each vertical line of the MIIPS trace corresponds to a SHG spectrum generated at given value of $\delta$ (redder colors represent higher intensities). For transform-limited pulses (b), the features form parallel lines separated by $\pi$. 
Kerr Effect

• Non linear refractive Index

\[ n = n_0 + n_2 I(t) \]

• Phase shift from propagation through medium of length \( L \)

\[ \Delta \phi(t) = \frac{-\omega}{c} n_2 I(t)L = \frac{-2\pi}{\lambda} n_2 I(t)L \]
Kerr Effect

- Nonlinear index proportional to $3^{rd}$ order susceptibility

\[ n_2 = \frac{16\pi^2 \chi^{(3)}}{n_0 c} \]

\[ n = n_0 + n_2 I \]

\[ \delta \phi = \frac{2\pi}{\lambda} n_2 I_0 \exp \left( -2 \frac{r^2}{w^2} \right) l \approx \frac{2\pi}{\lambda} n_2 I_0 \left( 1 - 2 \frac{r^2}{w^2} \right) \]

- Comparing to phase shift of thin lens:

\[ \delta \phi_{\text{lens}} = \frac{2\pi}{\lambda} n \frac{r^2}{f} \]
Kerr Effect – Focusing

- Focal length of effective lens is given by

\[ f = \frac{w^2}{2n_2 I_0 l} = \frac{\pi w^4}{4n_2 pl} \]
Self Starting

• It can be shown that we require

\[ \frac{\gamma}{g} > \frac{\sigma_g \tau_f}{\hbar \omega A_g} \]

\( \sigma_g = \text{gain cross section} \)
\( A_g = \text{Beam area in gain medium} \)
\( \tau_f = \text{duration of power fluctuation} \)
\( \gamma = \text{SAM proportionality constant} \)

• This is often not satisfied in solid state KLM lasers!
  – Unless we tap on optical table to reduce \( \tau_f \)
Figure 2.17  Stability regions and calculated low-power beam diameters for a KLM resonator, where $2w_1$ and $2w_2$ are the beam diameters at mirrors $M_1$ and $M_2$ respectively, in Fig. 2.16(a). $R_1$ and $R_2$ refer to the equivalent radii of curvature of eq. (2.108). Cavity parameters are given in the text. Adapted from Fig. 2 of [55], with permission from Springer Science and Business Media.
Figure 2.16  (a) Internal lens model for analysis of a KLM laser resonator; (b) equivalent two-mirror resonator.
Stability Calculation

\[
\frac{1}{s_i'} + \frac{1}{s_i} = \frac{1}{f_i}
\quad \text{for} \quad i = 1, 2
\quad (2.107)
\]

and the equivalent radii of curvature\(^3\) are given by

\[
R_i^{(eq)} = \frac{-f_i^2}{s_i - f_i}
\quad \text{for} \quad i = 1, 2
\quad (2.108)
\]

If we also introduce a parameter \(\Delta\) which characterizes the range of stable operation, defined by

\[
d' = f_1 + f_2 + \Delta
\quad (2.109)
\]

we find that the equivalent distance \(t\) between the two equivalent curved mirrors is

\[
t = R_1^{(eq)} + R_2^{(eq)} + \Delta
\quad (2.110)\]
Stability Calculation

\[ 0 < \Delta < -R_2^{(eq)} \quad \text{or} \quad -R_1^{(eq)} < \Delta < -\left( R_1^{(eq)} + R_2^{(eq)} \right) \quad (2.111) \]

**Figure 2.17** Stability regions and calculated low-power beam diameters for a KLM resonator, where \(2w_1\) and \(2w_2\) are the beam diameters at mirrors \(M_1\) and \(M_2\) respectively, in Fig. 2.16(a). \(R_1\) and \(R_2\) refer to the equivalent radii of curvature of eq. (2.108). Cavity parameters are given in the text. Adapted from Fig. 2 of [55], with permission from Springer Science and Business Media.
Stability Calculation

- Critical Power for self focusing
  \[ P_{\text{crit}} = \frac{a_{\text{SF}} \lambda^2}{8\pi n n_2} \]

- Parameter characterizing beam size with respect to power
  \[ \delta_w = \frac{1}{w(0)} \frac{\partial w(0)}{\partial (P/P_{\text{crit}})} \]

- Maximum value for parameter
  \[ |\delta_w| \leq \frac{1}{4\sqrt{g_1 g_2 (1 - g_1 g_2)}} \]
Figure 2.18  Contour plots showing calculated values of $\delta_w$ as a function of spherical mirror spacing and laser rod position. (a) $s_1 = 50\,\text{cm}$, $s_2 = 110\,\text{cm}$; (b) $s_1 = 70\,\text{cm}$, $s_2 = 90\,\text{cm}$. Filled circles indicate resonator configurations for which self-sustaining hard-aperture KLM was observed experimentally, and dashed horizontal lines indicate stability limits. In both cases, $f_1 = f_2 = 5\,\text{cm}$ and the slit for hard-aperture KLM was placed in the short arm of the laser cavity. Adapted from [76].
Yb:KYW Oscillator
Yb:KYW Oscillator
Yb:YAG Thin disk laser

a)

b)

c)

3.2 cm
Dispersion Consideration

• Different wavelengths of light in general have different refractive indices
  – Pulses broaden when passing through dispersive material

• Two types of dispersion
  – Anomalous
  – Normal
Dispersion

4.1.1 Group Velocity Definition and General Dispersion Relations

Consider an input pulse $e_{\text{in}}(t)$ with spectrum $E_{\text{in}}(\omega)$. After passing through a dispersive system that adds spectral phase $\psi(\omega)$, the output pulse is given by

$$e_{\text{out}}(t) = \frac{1}{2\pi} \int d\omega \ E_{\text{in}}(\omega) e^{j\omega t} e^{j\psi(\omega)}$$

(4.1)

When the phase arises due to passage through a bulk dispersion medium of length $L$, we can write

$$\psi(\omega) = -\beta(\omega)L$$

(4.2)

where $\beta(\omega)$ is the propagation constant in the medium. $\psi$ and $\beta$ are commonly written in terms of their Taylor series expansions, as follows:

$$\psi(\omega) = \psi(\omega_0) + \frac{\partial \psi}{\partial \omega}(\omega - \omega_0) + \frac{1}{2} \frac{\partial^2 \psi}{\partial \omega^2}(\omega - \omega_0)^2 + \frac{1}{6} \frac{\partial^3 \psi}{\partial \omega^3}(\omega - \omega_0)^3 + \cdots$$

$$= \psi_0 + \psi_1(\omega - \omega_0) + \frac{\psi_2}{2}(\omega - \omega_0)^2 + \frac{\psi_3}{6}(\omega - \omega_0)^3 + \cdots$$

(4.3)

Delay  Chirp 3rd order Chirp
Figure 4.5  (a) Treacy grating pair: a pair of parallel gratings with normal separation $G$ and slant distance $b$; (b) geometry for evaluating the path-length difference for a finite beam diffracting from a grating. $\theta_D$ is the diffraction angle measured with respect to the grating normal.
Dispersion Consideration

Figure 4.6  Grating pair with internal lenses.
Dispersion Consideration

Figure 4.7  Prism sequence for dispersion control: (a) two-prism sequence; (b) four-prism sequence.
Dispersion Consideration

Figure 4.17  (a) Chirped mirror designed to provide negative dispersion; (b) double-chirped mirror. From [208]. Copyright © 1998, IEEE.
Dispersion Consideration

• For a mode-locked laser to produce stable pulses, the net dispersion must be nearly zero

• Soliton mode-locking
  – The competition between nonlinearities and dispersion produces a stable soliton pulse train
    • Usually interaction between self phase modulation and dispersion
  – Usually produced with negative dispersion cavities
Pulse Front Tilt

Figure 4.12 Radially varying pulse delay for a lens with chromatic aberration. Material dispersion is not included. From [195].
Kerr Effect – Self Phase Modulation

Figure 1: Instantaneous frequency of an initially unchirped pulse which has experienced self-phase modulation. The central part of the pulse exhibits an up-chirp.
Self Phase Modulation

- For pulse with Gaussian shape
  \[ I(t) = I_0 e^{-t^2/t_p^2} \]

- Nonlinear refractive index
  \[ n(I) = n_0 + n_2 I \]

- Produces time varying refractive index
  \[ \frac{dn(I)}{dt} = n_2 \frac{dI}{dt} = n_2 I_0 \frac{-2t}{\tau^2} \exp \left( \frac{-t^2}{\tau^2} \right) \]

- Time dependent phase shift
  \[ \phi(t) = \omega_0 t - k z = \omega_0 t - \frac{2\pi}{\lambda_0} n(t) L \]
Self Phase Modulation

• Time dependent instantaneous phase shift

\[ \phi(t) = \omega_0 t - kz = \omega_0 t - \frac{2\pi}{\lambda_0} n(t) L \]

• Frequency shift of the pulse is then

\[ \omega(t) = \frac{d\phi(t)}{dt} = \omega_0 - \frac{2\pi L}{\lambda_0} n(I) L \]

• Using expression for \( dn/dt \) we have

\[ \omega(t) = \omega_0 + \frac{4\pi Ln_2 I_0}{\lambda_0 \tau^2} t \exp \left( \frac{-t^2}{\tau^2} \right) \]

Positive Chirp
Self Phase Modulation

- Leading edge of pulse is red-shifted
- Trailing edge is blue-shifted

A pulse (top curve) propagating through a nonlinear medium undergoes a self-frequency shift (bottom curve) due to self-phase modulation. The front of the pulse is shifted to lower frequencies, the back to higher frequencies. In the centre of the pulse the frequency shift is approximately linear.
Setups we have worked on

- Yb:KYW oscillator
- Ti:Sapphire
- To come:
  - Yb:YAG thin disk oscillator
Figure 3.2  Michelson interferometer setup for measuring the electric field autocorrelation function.
• Linear Autocorrelation is not sensitive to phase!

Figure 3.3 Example of time-average interferometer output power as a function of delay $\tau$. The power is normalized by the factor $\frac{1}{2}\Gamma_a(0)$. This figure corresponds to a Gaussian input pulse with $t_p = 20$ fs and an optical period equal to 5 fs.
Intensity Autocorrelation

Figure 3.8  Collinear second-harmonic generation geometry for measurement of the intensity autocorrelation function. PMT, photomultiplier tube.
Noncollinear Intensity Autocorrelation

Figure 3.9 Noncollinear second-harmonic generation geometry for background-free intensity autocorrelation function measurement. PMT, photomultiplier tube.
Figure 3.11 (a) Intensity autocorrelation and (b) interferometric autocorrelation for nearly bandwidth-limited pulses from a mode-locked Ti:S laser. A 60-fs pulse width and a time–bandwidth product $\Delta \nu \Delta t = 0.33$ are calculated assuming a $\text{sech}^2$ intensity profile. For (b), the delay axis is zoomed by roughly a factor of 3 compared to (a). From [52].
Table 3.2  Autocorrelation Functions and Deconvolution Factors for Four Pulse Shapes

<table>
<thead>
<tr>
<th>$I(t)$</th>
<th>$G_2(\tau)$</th>
<th>$\frac{\Delta \tau}{\Delta t}$</th>
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<tbody>
<tr>
<td>sq $(t)$</td>
<td>$1 -</td>
<td>\tau</td>
</tr>
<tr>
<td>$e^{-2t^2}$</td>
<td>$e^{-\tau^2}$</td>
<td>$\sqrt{2}$</td>
</tr>
<tr>
<td>sech² $(t)$</td>
<td>$\frac{3 (\tau \cosh(\tau) - \sinh(\tau))}{\sinh^3(\tau)}$</td>
<td>1.543</td>
</tr>
<tr>
<td>$e^{-t}$ for $t \geq 0$, 0 otherwise</td>
<td>$e^{-</td>
<td>\tau</td>
</tr>
</tbody>
</table>

$\Delta t$ and $\Delta \tau$ are full widths at half maximum of the intensity $I(t)$ and the intensity autocorrelation function $G_2(\tau)$, respectively. sq$(u)$ is defined as a unit square pulse such that sq$(u) = 1$ for $|u| \leq \frac{1}{2}$ and 0 otherwise; and $t$ and $\tau$ are in normalized units.
Figure 3.12  (a) Intensity autocorrelation and (b) interferometric autocorrelation for highly chirped pulses from a mode-locked Ti:S laser. A 2-ps pulse width and a time–bandwidth product $\Delta v \Delta t = 2.7$ are calculated assuming a sech$^2$ intensity profile. For (b), the delay axis is zoomed by roughly a factor of 3 compared to (a). From [52].
Frequency-Resolved Optical Gating (FROG)

SHG FROG is the most sensitive version of FROG.
Applications of mode-locked lasers

- Ultrafast spectroscopy
- Laser-controlled chemistry
- Frequency metrology
- High-speed electrical testing
- Laser-plasma interactions
- Short wavelength generation
- Optical communications
- Materials processing
- Biomedical applications