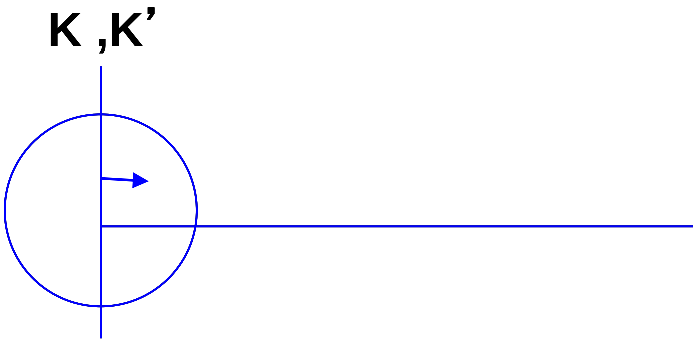
Derivation of Lorentz Transformations

- Use the fixed system K and the moving system K'
- *At t* = 0 the origins and axes of both systems are ***coincident*** with system K' moving to the right along the *x* axis.
- A flashbulb goes off at the origins when *t* = 0.
- According to postulate 2, the speed of light will be *c* in both systems and the wavefronts observed in both systems must be spherical.



Derivation (con't)

Spherical wavefronts in K:

$$x^2 + y^2 + z^2 = c^2t^2$$

Spherical wavefronts in K':

$$x'^2 + y'^2 + z'^2 = c^2t'^2$$

Note: these are not preserved in the classical transformations with

$$x' = x - vt$$

$$y' = y$$

$$z' = z$$

$$t = t'$$

Derivation (con't)

1) Let $x' = \gamma (x - vt)$ so that $x = \gamma' (x' + vt')$

2) By Einstein's first postulate: $\gamma = \gamma'$

3) The wavefront along the x,x' - axis must satisfy:

   $x = ct$ and $x' = ct'$

4) Thus $ct' = \gamma (ct - vt)$ and $ct = \gamma (ct' + vt')$

5) Solving the first one above for t' and substituting into the second...

Derivation of the Lorentz transformation

The simplest linear transformation

$$x' = \gamma(x - vt) \qquad x = ct$$

$$x = \gamma'(x' + vt') \qquad x' = ct'$$

$$\gamma' = \gamma$$

Principle of relativity

Consider expanding light is spherical, then light travels a distance

$$ct' = \gamma(ct - vt)$$

$$ct = \gamma(ct' + vt')$$

Divide each equation by c

$$t' = \gamma t(1 - \frac{v}{c})$$

$$t = \gamma t'(1 + \frac{v}{c})$$

Substitute t from the lower to the upper equation

$$t' = \gamma^2 t'(1 - \frac{v^2}{c^2})$$

Solve for γ^2

$$\gamma^2 = \frac{1}{1 - v^2/c^2} \rightarrow \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Find transformation for the time t'

We had

$$x = \gamma'(x' + vt')$$

$$x' = \gamma(x - vt)$$

$$\gamma = \gamma'$$

$$t' = \gamma t\left(1 - \frac{v}{c}\right)$$

$$t = \frac{x}{c}$$

t=>x/c

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c}}}$$

The complete Lorentz Transformations Including the inverse (i.e v replaced with –v; and primes interchanged)

$$x' = \frac{x - vt}{\sqrt{1-\beta^2}} \qquad x = \frac{x' + vt'}{\sqrt{1-\beta^2}}$$

$$y' = y \qquad y = y'$$

$$z' = z \qquad z = z'$$

$$t' = \frac{t - (vx/c^2)}{\sqrt{1-\beta^2}} \qquad t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1-\beta^2}}$$

2.4. # 11. Show that both Eqs. (2.17) and (2.18) reduce to the Galilean transformation when v<<c.

When $v << c$ we find $1 - \beta^2 \rightarrow 1$, so

Eqs. (2.17)

$$x' = \frac{x - \beta ct}{\sqrt{1-\beta^2}} \rightarrow x - \beta ct = x - vt$$

$$t' = \frac{t - \beta x/c}{\sqrt{1-\beta^2}} \rightarrow t - \beta x/c \approx t$$

Eqs. (2.18)

$$x = \frac{x' + \beta ct'}{\sqrt{1-\beta^2}} \rightarrow x' + \beta ct' = x' + vt'$$

$$t = \frac{t' + \beta x'/c}{\sqrt{1-\beta^2}} \rightarrow t' + \beta x'/c \approx t'$$

Remarks

1) If $v << c$, i.e., $\beta \approx 0$ and $\gamma \approx 1$, we see these equations reduce to the familiar Galilean transformation.

2) Space and time are now not separated.

3) For non-imaginary transformations (which is required to have physical sense), the frame velocity cannot exceed *c*.

(Note: values are somewhat changed compared to #12)

12. Determine the ratio $\beta = v/c$ for the following: (a) A car traveling 100 km/h. (b) A commercial jet airliner traveling 290 m/s. (c) A supersonic airplane traveling at Mach 2.3 (Mach number = v/v_{sound}). (d) The space station, traveling 27,000 km/h. (e) An electron traveling 25 cm in 2 ns. (f) A proton traveling across a nucleus (10^{-14} m) in 0.35×10^{-22} s.

a) Conversion 100 km/h = 27.77 m/s so

$$\beta = v/c = (27.77\ \text{m/s})/\left(3.00 \times 10^8\ \text{m/s}\right) = 9.3 \times 10^{-8}$$

b) $\beta = v/c = (290\ \text{m/s})/\left(3.00 \times 10^8\ \text{m/s}\right) = 9.7 \times 10^{-7}$

c) $v = 2.3 v_{\text{sound}} = (2.3 \cdot 330\ \text{m/s})$ and

$$\beta = v/c = (2.3 \cdot 330\ \text{m/s}) / \left(3.00 \times 10^8\ \text{m/s}\right) = 2.5 \times 10^{-6}$$

d) Conversion 27,000 km/h = 7500 m/s so

$$\beta = v/c = (7500\text{m/s})/\left(3.00 \times 10^8 \text{m/s}\right) = 2.5 \times 10^{-5}$$

e) $(25\text{cm}) / (2\text{ns}) = 1.25 \times 10^8\ \text{m/s}$ so $\beta = v/c = (1.25 \times 10^8\ \text{m/s})/\left(3.00 \times 10^8\ \text{m/s}\right) = 0.42$

f) $\left(1 \times 10^{-14}\text{m}\right) / \left(0.35 \times 10^{-22}\text{s}\right) = 2.857 \times 10^8\ \text{m/s}$

$$\beta = v/c = (2.857 \times 10^8\ \text{m/s})/\left(3.00 \times 10^8\ \text{m/s}\right) = 0.95$$

2.4
#13

Two events occur in an inertial system K as follows:

Event 1: $x_1 = a, \quad t_1 = 2a/c, y_1 = 0, z_1 = 0$
Event 2: $x_2 = 2a, \quad t_2 = 3a/2c, y_2 = 0, z_2 = 0$

In what frame K′ will these events appear to occur at the same time? Describe the motion of system K′.

$$t'_1 = \frac{t_1 - \frac{vx_1}{c^2}}{\sqrt{1 - \frac{v^2}{c}}} \qquad t'_2 = \frac{t_2 - \frac{vx_2}{c^2}}{\sqrt{1 - \frac{v^2}{c}}}$$

We require $t'_1 = t'_2$

$$t_1 - \frac{vx_1}{c^2} = t_2 - \frac{vx_2}{c^2}$$

Plugging values for Events 1 and 2 and solving the equation for v, velocity of K’ relative to K, we find v= - c/2.

2.5: Time Dilation and Length Contraction

Consequences of the Lorentz Transformation:

- **Time Dilation**:

  Clocks in K' run slow with respect to stationary clocks in K.

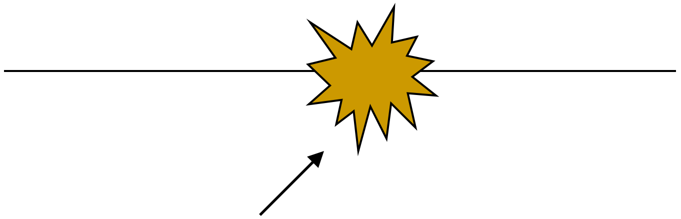
- **Length Contraction**:

  Lengths in K' are contracted with respect to the same lengths stationary in K.

Time Dilation

To understand *time dilation* the idea of **proper time** must be understood:

- The term **proper time**, T_0, is the time difference between two events occurring at the same position in a system as measured by a clock at that position.



Same location (spark "on" then off")

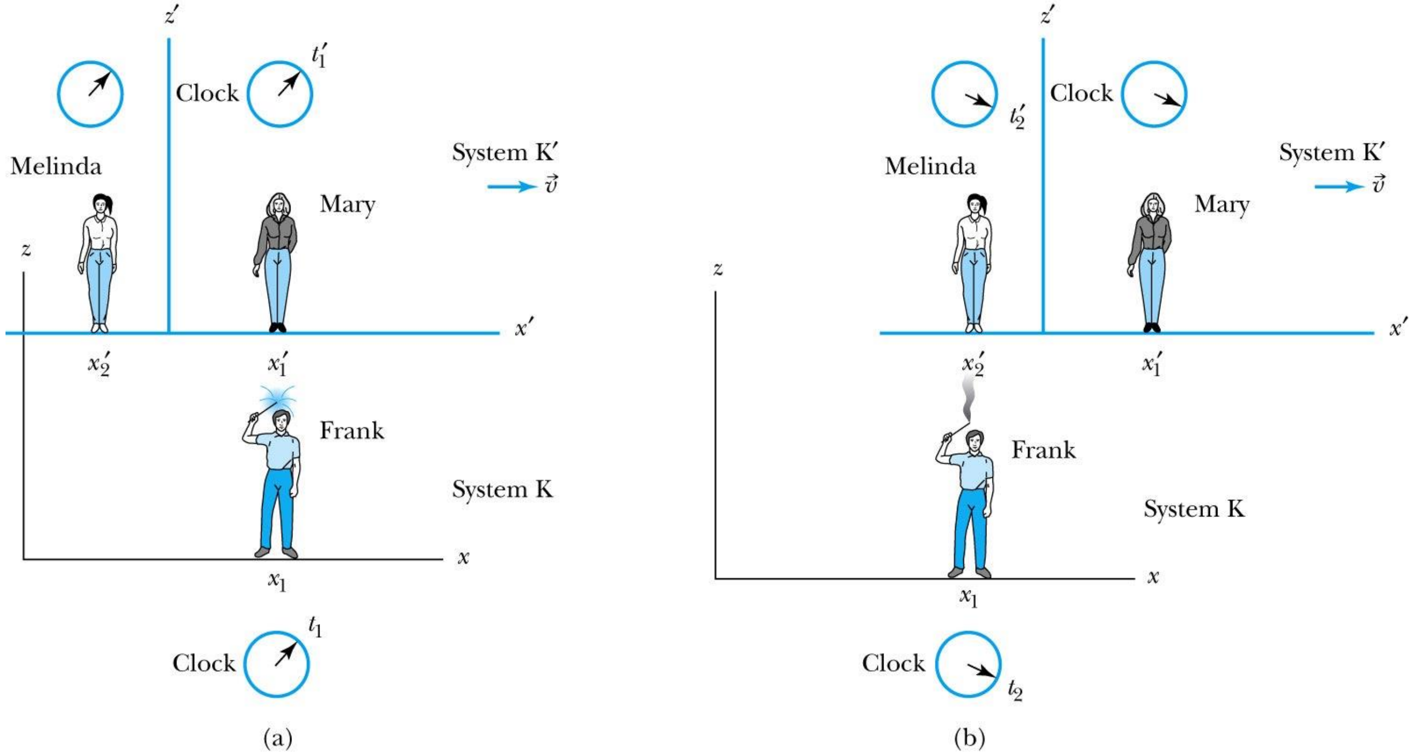
Time Dilation

Not Proper Time

spark "on" then spark "off"

Beginning and ending of the event occur at different positions

Time Dilation with Mary, Frank, and Melinda



Frank's clock is at the same position in system K when the sparkler is lit in (a) and when it goes out in (b). Mary, in the moving system K', is beside the sparkler at (a). Melinda then moves into the position where and when the sparkler extinguishes at (b). Thus, Melinda, at the new position, measures the time in system K' when the sparkler goes out in (b).

According to Mary and Melinda…

- Mary and Melinda measure the two times for the sparkler to be lit and to go out in system K' as times t'_1 and t'_2 so that by the Lorentz transformation:

$$t'_2 - t'_1 = \frac{(t_2 - t_1) - (v/c^2)(x_2 - x_1)}{\sqrt{1 - v^2/c^2}}$$

  - Note here that Frank records $x_2 - x_1 = 0$ in K with a proper time: $T_0 = t_2 - t_1$ or

$$T' = \frac{T_0}{\sqrt{1 - v^2/c^2}} = \gamma T_0$$

with $T' = t'_2 - t'_1$

Time Dilation:

Moving Clocks Run Slow

1) $T' > T_0$ or the time measured between two events in moving system *K'* is greater than the time between the same events in the system K, where they are at rest: ***time dilation.***
2) The events do not occur at the same space and time coordinates in the two systems
3) System K requires 1 clock and K' requires 2 clocks.

Length Contraction

To understand *length contraction* the idea of **proper length** must be understood:

- Let an observer in each system K and K' have a meter stick at rest in ***their own system*** such that each measure**s** the same length at rest.
- The length as measured at rest is called the **proper length**.

What Frank and Mary measure in their own reference frames

Each observer lays the stick down along his or her respective *x* axis, putting the left end at x_ℓ (or x'_ℓ) and the right end at x_r (or x'_r).

- Thus, in system K, Frank measures his stick to be:

$$L_0 = x_r - x_\ell$$

- Similarly, in system K', Mary measures her stick at rest to be:

$$L'_0 = x'_r - x'_\ell = L_0$$

What Frank and Mary measure for a moving stick

- Frank in his rest frame measures the length of the stick for Mary's frame moving with relative velocity.

- Thus, according to the Lorentz transformations :

$$x'_r - x'_\ell = \frac{(x_r - x_\ell) - v(t_r - t_\ell)}{\sqrt{1 - v^2/c^2}}$$

It is assumed that both ends of the stick are measured simultaneously, i.e, $t_r = t_\ell$ and =>t_r - t_ℓ=0

Here Mary's proper length is $L'_0 = x'_r - x'_\ell$

and Frank's measured length is $L = x_r - x_\ell$

Frank's measurement

So Frank measures the moving length as *L* given by

$$L'_0 = \frac{L}{\sqrt{1 - v^2/c^2}} = \gamma L$$

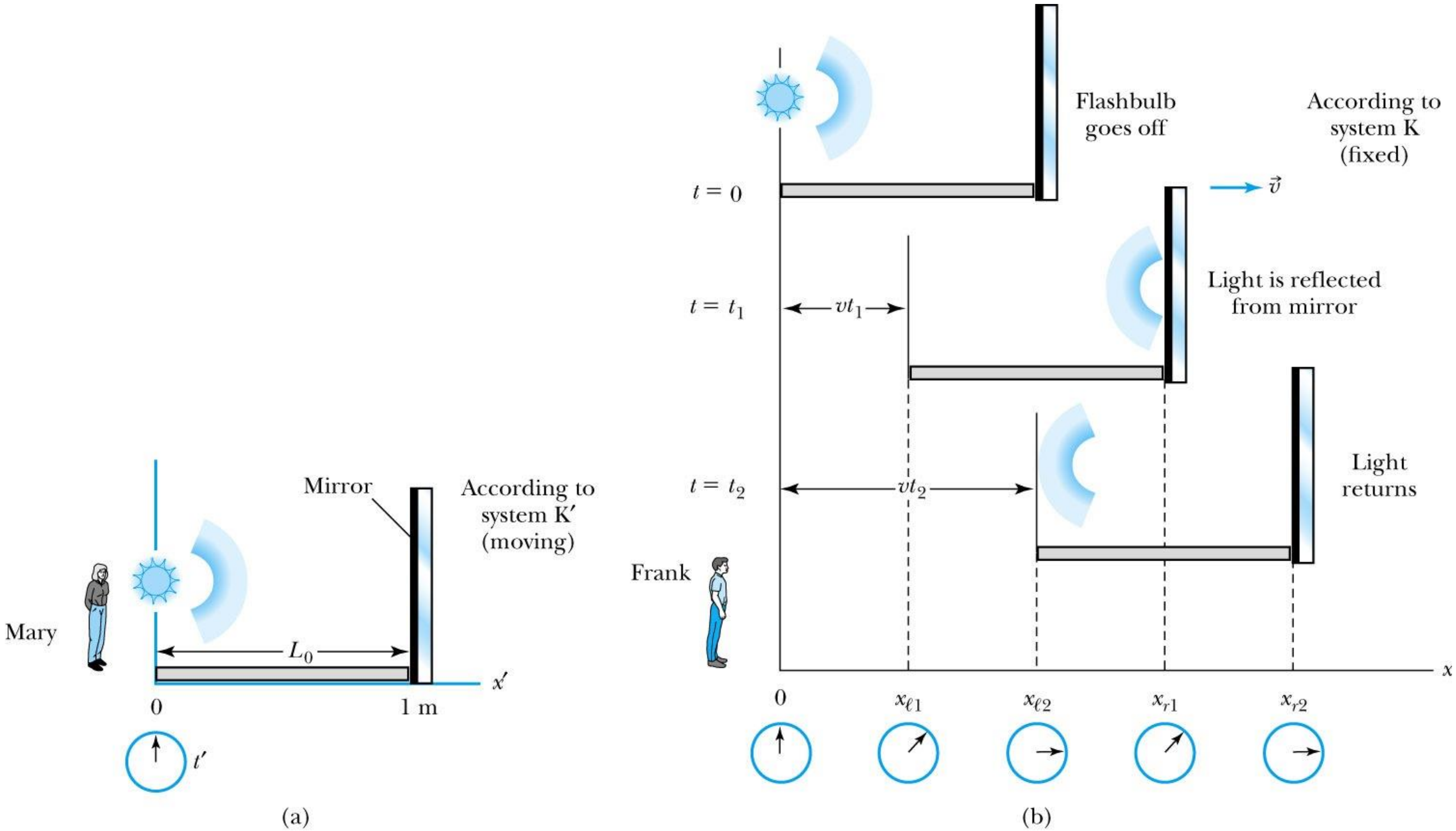
but since both Mary and Frank in their respective frames measure $L'_0 = L_0$

i.e. the measured length for the moving stick shrinks

$$L = L_0\sqrt{1 - v^2/c^2} = \frac{L_0}{\gamma}$$

and $L_0 > L$.

A “Gedanken Experiment” to Clarify Length Contraction



2.5
#18

Show that the experiment depicted in Figure 2.11 and discussed in the text leads directly to the derivation of length contraction.

At the point of reflection the light has traveled a distance

$$L + v\Delta t_1 = c\Delta t_1$$

on the return it travels

$$L - v\Delta t_2 = c\Delta t_2$$

→ total round trip time

$$\Delta t = \Delta t_1 + \Delta t_2$$

$$= \frac{L}{c-v} + \frac{L}{c+v} = \frac{Lc + vL + cL - vL}{c^2 - v^2}$$

$$= \frac{2Lc}{c^2 - v^2} = \frac{2L/c}{1 - v^2/c^2}$$

but from time dilation we know (with $\Delta t' =$ proper time $= 2L_0/c$) that

$$\Delta t = \gamma \Delta t' = \frac{2L_0/c}{\sqrt{1 - v^2/c^2}}$$

comparing the two results for Δt we get

$$\frac{2L/c}{1 - \frac{v^2}{c^2}} = \frac{2L_0/c}{\sqrt{1 - \frac{v^2}{c^2}}}$$

which reduces to

$$L = L_0\sqrt{1 - v^2/c^2} = \frac{L_0}{\gamma}$$

Example

2.5
#22

The Apollo astronauts returned from the moon under the Earth's gravitational force and reached speeds of almost 25,000 mi/h with respect to Earth. Assuming (incorrectly) they had this speed for the entire trip from the moon to Earth, what was the time difference for the trip between their clocks and clocks on Earth?

2.5
#22

Converting the speed to m/s we find 25000 mi/h = 11,176 m/s

distance$_{\text{eart-moon}}$ = 3.84×10^8 m

In the earth frame $t = \frac{d}{v} = \frac{3,84 \times 10^8 \text{ m}}{11,176 \text{ m/s}} = 34,359 \text{ s}$

In the astronaut frame $t' = \frac{t}{\gamma} = t\sqrt{1-\beta^2}$

The time difference $\Delta t = t - t' = t - t\sqrt{1-\beta^2}$

$= t(1 - \sqrt{1-\beta^2})$

$= 34,359 \text{ s} \left[1 - \sqrt{\left(\frac{11,176 \text{ m/s}}{3 \times 10^8 \text{ m/s}}\right)^2 + 1}\right]$

$\Delta t = 2.4 \times 10^{-5} \text{ s}$

Time dilation $t' < t$

2.5
#28

Imagine that in another universe the speed of light is only 100 m/s. (a) A person traveling along an interstate highway at 120 km/h ages at what fraction of the rate of a person at rest? (b) This traveler passes by a meterstick at rest on the highway. How long does the meterstick appear?

(a) Converting $v = 120$ km/h $= 33.3$ m/s. Now with $c = 100$ m/s we have $\beta = \frac{v}{c} = 0.333$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-0.333^2}} = 1.061$$

We conclude that the moving person ages 6.1 % slower

(b) $$L' = \frac{L}{\gamma} = \frac{1\,m}{1.061} = 0.942\,m$$

Problem 100, ch.2

The Lockheed SR-71 Blackbird may be the fastest non-research airplane ever built; it traveled at 2200 miles/hour (983 m/s) and was in operation from 1966 to 1990. Its length is 32.74 m. (a) By what percentage would it appear to be length contracted while in flight? (b) How much time difference would occur on an atomic clock in the plane compared to a similar clock on Earth during a flight of the Blackbird over its range of 3200 km?

$\beta = \frac{v}{c} = \frac{983 \text{ m/s}}{3\times10^8 \text{ m/s}} = 3.28\times10^{-6}$ which is very small, so we use the binomial approximation

Length contraction $L = \frac{L_0}{\gamma}$ $\quad \gamma^{-1} = \sqrt{1-\beta^2} \simeq 1 - \frac{1}{2}\beta^2$ $\quad$ note: $(1-x)^{+\frac{1}{2}} = 1 - \frac{1}{2}x + \ldots$; $(1-x)^{-\frac{1}{2}} = 1 + \frac{1}{2}x + \ldots$ for $x \ll 1$

(a) The percentage of length contraction is

$$\% \text{change} = \frac{L_0 - L}{L_0}\times 100\% = \frac{L_0 - L_0\gamma^{-1}}{L_0}\times100\% = [1-\gamma^{-1}]\times100\%$$

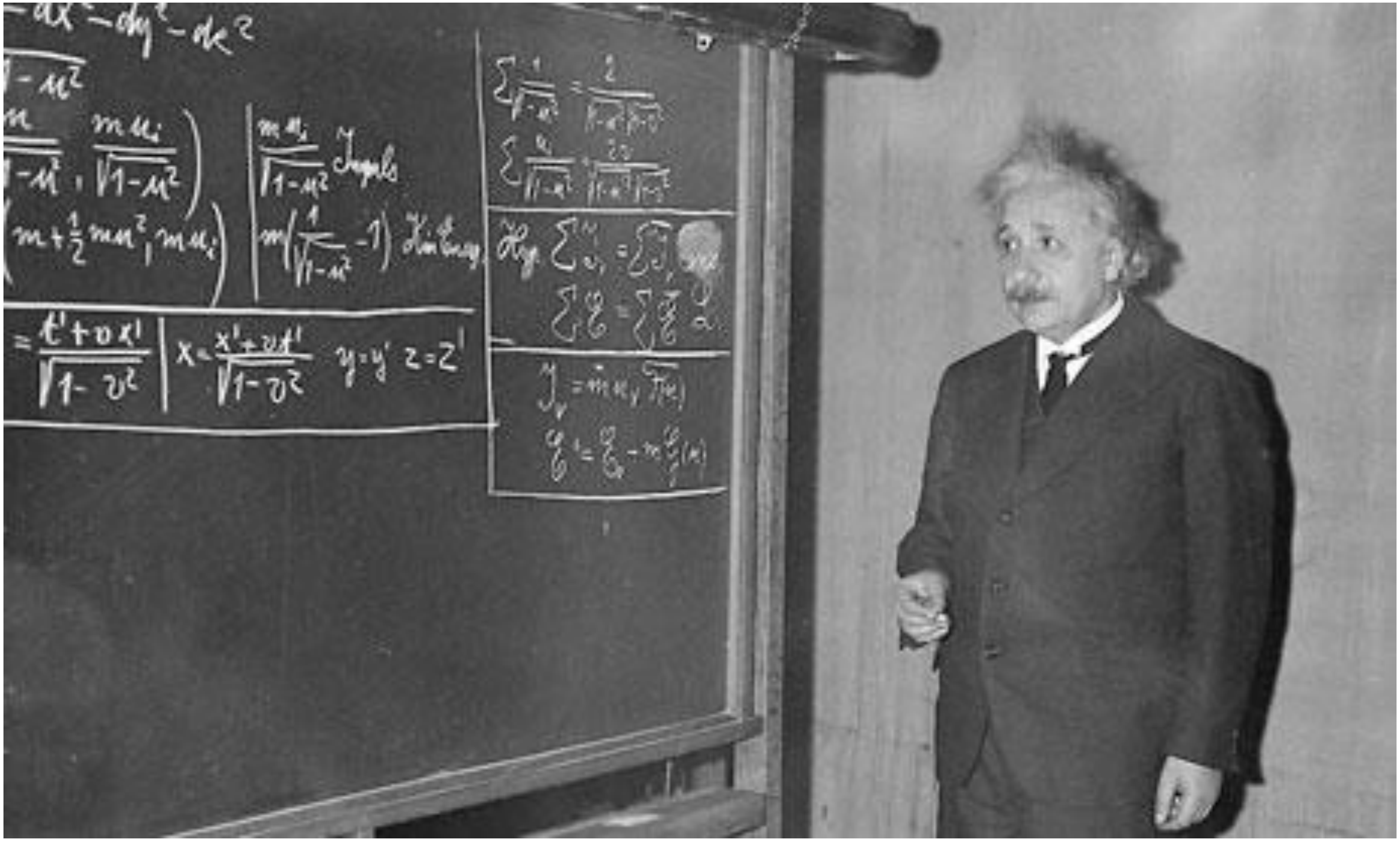
$$= [1 - (1 - \tfrac{1}{2}\beta^2)]\times100\% = \tfrac{1}{2}\beta^2\times100\% = \boxed{5.37\times10^{-10}}$$

b) The clocks' rates differ by $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. The clock on the SR-71 measures the proper time

Time dilation $T' = \gamma T_0$ $\Rightarrow$ $\Delta t = T' - T_0 = \gamma T_0 - T_0 = T_0(\gamma - 1) = 1 + \frac{1}{2}\beta^2 - 1 = \frac{1}{2}\beta^2$

$$\Delta t = \underbrace{\frac{3.2\times10^6 \text{ m}}{983 \text{ m}}}_{T_0} \cdot \frac{\left(\frac{983 \text{ m}}{3\times10^8 \text{ m}}\right)^2}{2} = 1.75\times10^{-8}\text{ s} = \boxed{17.5 \text{ ns}}$$

Albert Einstein lecturing on the special theory of relativity. Photograph: AP



2.6: Addition of Velocities

Taking differentials of the Lorentz transformation, relative velocities may be calculated:

$$dx = \gamma(dx' + v\,dt')$$

$$dy = dy'$$

$$dz = dz'$$

$$dt = \gamma[dt' + (v/c^2)\,dx']$$

So that…

defining velocities as: $u_x = dx/dt$, $u_y = dy/dt$, $u'_x = dx'/dt'$, etc. it is easily shown that:

$$u_x = \frac{dx}{dt} = \frac{\gamma(dx' + v\,dt')}{\gamma[dt' + (v/c^2)\,dx']} = \frac{u'_x + v}{1 + (v/c^2)u'_x}$$

With similar relations for u_y and u_z:

$$u_y = \frac{u'_y}{\gamma\left[1 + (v/c^2)u_x'\right]} \qquad u_z = \frac{u'_z}{\gamma\left[1 + (v/c^2)u_x'\right]}$$

The Lorentz Velocity Transformations

In addition to the previous relations, the **Lorentz velocity transformations** for u'_x, u'_y, and u'_z can be obtained by switching primed and unprimed and changing v to $-v$:

$$u'_x = \frac{u_x - v}{1 - (v/c^2)u_x}$$

$$u'_y = \frac{u_y}{\gamma\left[1 - (v/c^2)u_x\right]}$$

$$u'_z = \frac{u_z}{\gamma\left[1 - (v/c^2)u_x\right]}$$

2.7: Experimental Verification

Time Dilation and Muon Decay

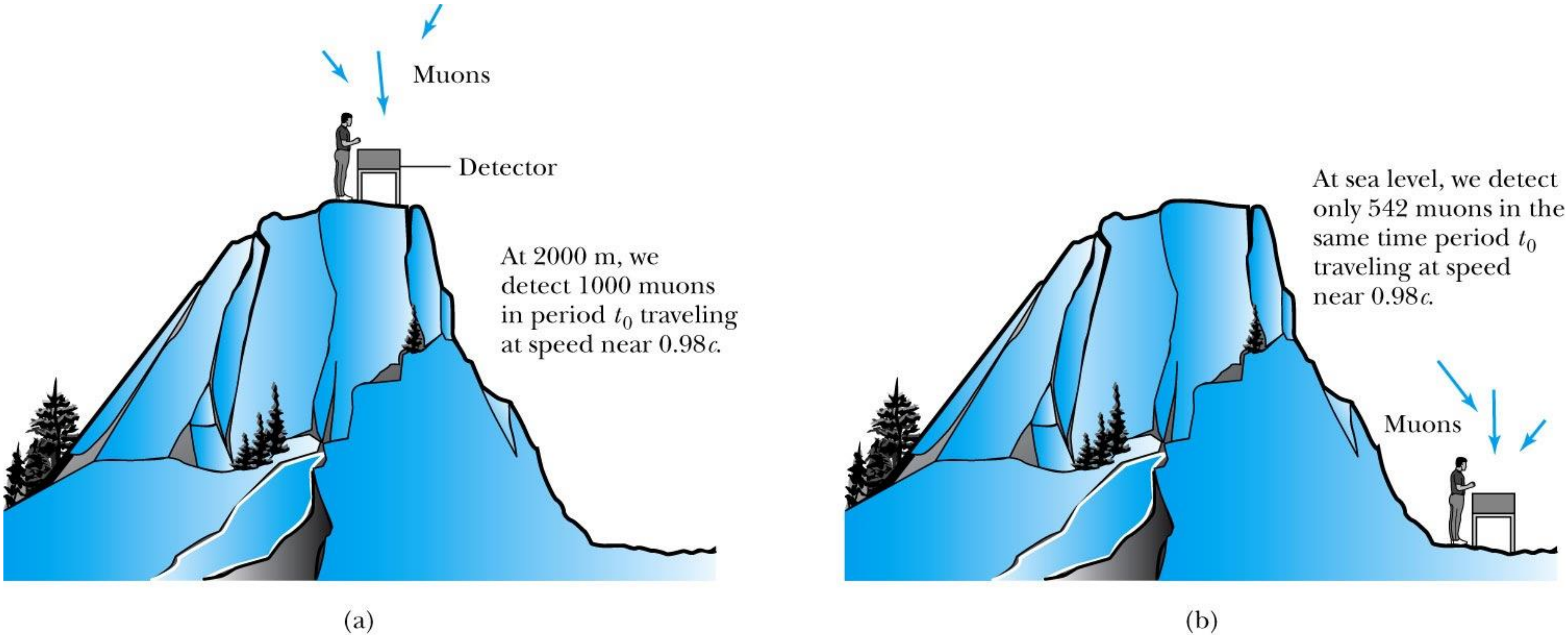


Figure 2.18: The number of muons detected with speeds near 0.98*c* is much different (a) on top of a mountain than (b) at sea level, because of the muon's decay. The experimental result agrees with our time dilation equation.

Muon decay (experimental verification of time dilation)

radioactive decay $N = N_0 e^{-\frac{\ln(2) t}{t_{1/2}}} = N_0 e^{-\frac{0.693 t}{t_{1/2}}}$ $\boxed{t_{1/2}(\text{muon}) = 1.52 \times 10^{-6} s}$

experiment with a system that moves close to c, namely $v = 0.98c$
put a muon detector on top of mount Wilson (2000m), then bring the detector to sea level ($h = 0$ m). Assume average muon flux is the same

$N_0^{(2000)} = 1000$ $v = 0.98c$
$N(0) = 540$

classically: $s = v \cdot t$ $t = \frac{s}{v} = \frac{2000}{0.98 \times 3 \times 10^8} = 6.8 \times 10^{-6} s$

$N(0) = 1000\, e^{-\frac{0.693 \times 6.8 \times 10^{-6}}{1.52 \times 10^{-6}}} = 45$ survive but experiment shows 540!

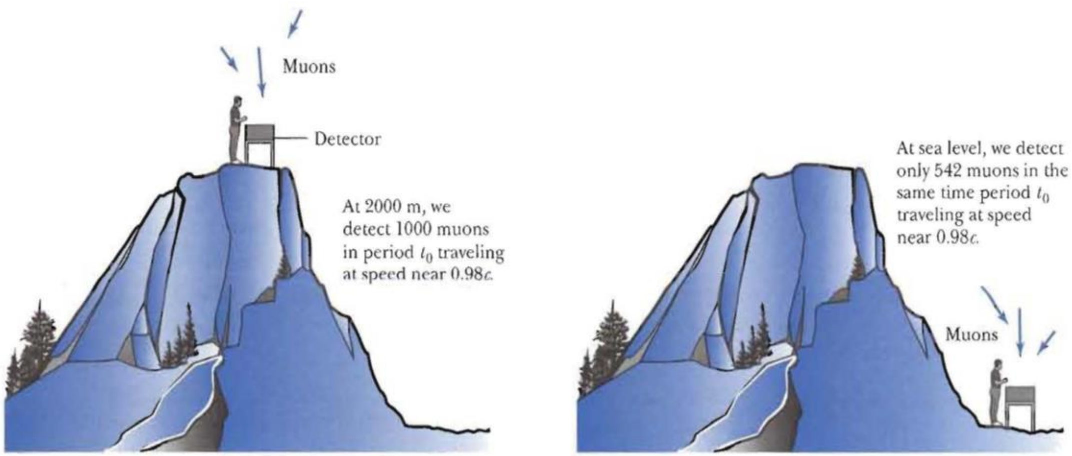
relativistically: $T' = \gamma T_0$
$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - \frac{0.98^2 c^2}{c^2}}} = 5$

In the muon restframe the time period for muons to travel 2000m (as seen by a clock fixed to the mountain)?

$\frac{T'}{\gamma} = \frac{6.8 \times 10^{-6}}{5} = 1.36 \times 10^{-6} s$

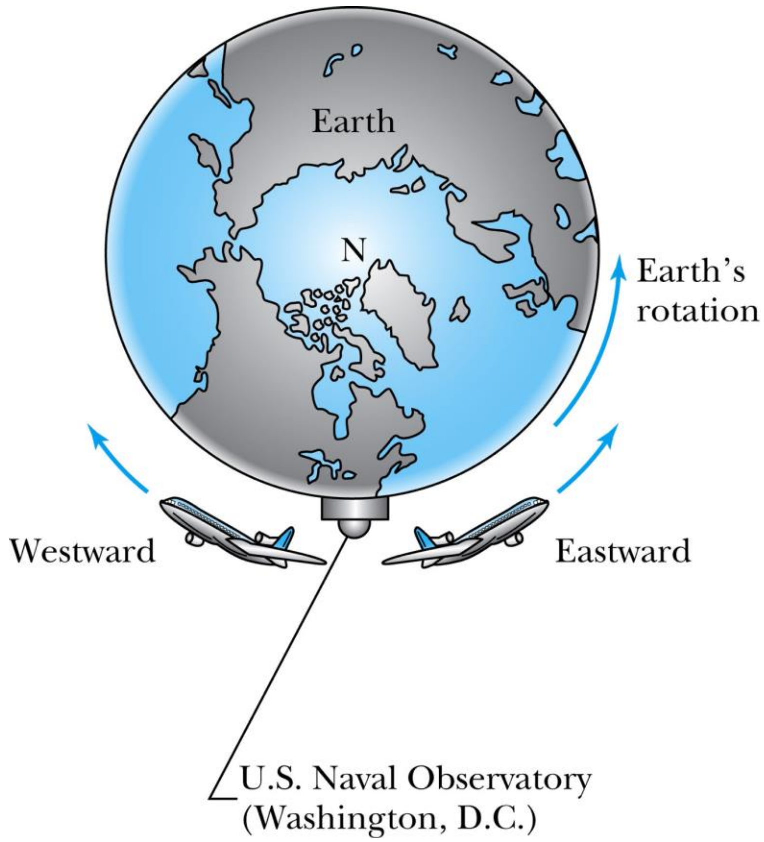
$N(0) = 1000 \times e^{-\frac{0.693 \times 1.36 \times 10^{-6}}{1.52 \times 10^{-6}}} = 538$

same as experiment



(a) (b)

Atomic Clock Measurement

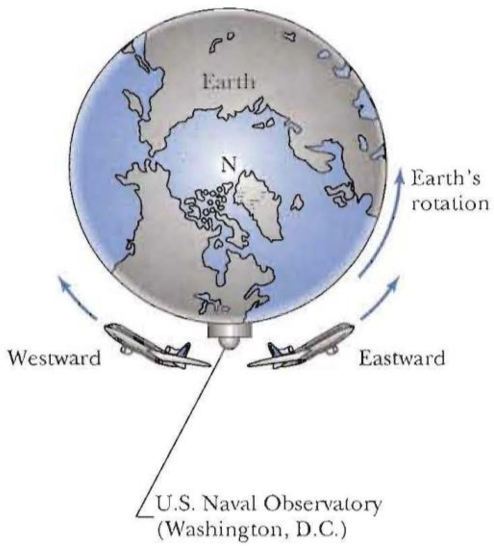


Two airplanes took off (at different times) from Washington, D.C., where the U.S. Naval Observatory is located. The airplanes traveled east and west around Earth as it rotated. Atomic clocks on the airplanes were compared with similar clocks kept at the observatory to show that the moving clocks in the airplanes ran differently.

Atomic Clock Measurement

^{133}Cs atom ($f \approx 9.2 \ldots$ GHz ; 1 GHz $= 10^9$ Hz) H-maser ; Hg^+-stored ion standard

test time dilation by flying cesium clocks around the world in commercial jet-liners (1971) and comparing them by a "stationary" reference clock at NBS and the Naval observatory in Washington DC.



Travel	Predicted	Observed	Flight time
Eastward	−40 ± 23 ns	−59 ± 10 ns	(41.2 h)
Westward	275 ± 21 ns	273 ± 7 ns	(48.6 h)

A negative time is less than reference clock = run slower = time dilation
A positive time is more than reference clock = run faster!

The time is changing in the moving frame, but the calculations must also take into account corrections due to general relativity (Einstein). Analysis shows that the special theory of relativity is verified within the experimental uncertainties.

Respondus lockdown browser

https://www.youtube.com/watch?v=XuX8WoeAycs&feature=youtu.be

https://web.respondus.com/he/monitor/resources/