CHAPTER 15

General Relativity

- 15.1 Tenets of General Relativity
- 15.2 Tests of General Relativity
- 15.3 Gravitational Waves
- 15.4 Black Holes and Neutron Stars

Time and space and gravitation have no separate existence from matter.
Albert Einstein
Comprehensive overview of the Solar System. The Sun, planets, dwarf planets and moons are at scale for their relative sizes, not for distances. A separate distance scale is at the bottom. Moons are listed near their planets by proximity of their orbits; only the largest moons are shown.
Merging Black Holes
15.1: Tenets of General Relativity

- General relativity is the extension of special relativity. It includes the effects of accelerating objects and their mass on spacetime.
- As a result, the theory is an explanation of gravity.
- It is based on two concepts: (1) the principle of equivalence, which is an extension of Einstein’s first postulate of special relativity and (2) the curvature of spacetime due to gravity.
General theory of relativity: Principle of equivalence

\[ \vec{F} = m_I \vec{a} \]
\[ \vec{F}_G = m_G \vec{a} \]
\[ \vec{a} = \frac{m_G}{m_I} \vec{g} \]
\[ m_G = m_I \]

(proved to 1 in \(10^{12}\))

Principle of equivalence: There is no experiment that can be done in a small confined space that can detect the difference between a uniform gravitational field and an equivalent uniform acceleration.
Bending of light: acceleration, gravitation

According to the astronaut inside the spacecraft, the light curved downward and must have been affected by the acceleration.

Due to the equivalence principle, the same thing must happen on the Earth because of gravity.
Distortions of space-time by a mass

Mass-energy tells spacetime how to curve.

Spacetime curvature tells matter how to move.
15.2: Tests of General Relativity

Bending of Light

- During a solar eclipse of the sun by the moon, most of the sun’s light is blocked on Earth, which afforded the opportunity to view starlight passing close to the sun in 1919. The starlight was bent as it passed near the sun which caused the star to appear displaced.

- Einstein’s general theory predicted a deflection of 1.75 seconds of arc, and the two measurements found $1.98 \pm 0.16$ and $1.61 \pm 0.40$ seconds.

- Since the eclipse of 1919, many experiments, using both starlight and radio waves from quasars, have confirmed Einstein’s predictions about the bending of light with increasingly good accuracy.
solar eclipse of the sun by the moon
Lunar eclipse
Gravitational Lensing

- When light from a distant object like a quasar passes by a nearby galaxy on its way to us on Earth, the light can be bent multiple times as it passes in different directions around the galaxy.
Gravitational Redshift

- The second test of general relativity is the predicted frequency change of light near a massive object.
- Imagine a light pulse being emitted from the surface of the Earth to travel vertically upward. The gravitational attraction of the Earth cannot slow down light, but it can do work on the light pulse to lower its energy. This is similar to a rock being thrown straight up. As it goes up, its gravitational potential energy increases while its kinetic energy decreases. A similar thing happens to a light pulse.
- A light pulse’s energy depends on its frequency $f$ through Planck’s constant, $E = hf$. As the light pulse travels up vertically, it loses kinetic energy and its frequency decreases. Its wavelength increases, so the wavelengths of visible light are shifted toward the red end of the visible spectrum.
- This phenomenon is called gravitational redshift.
Gravitational Redshift Experiments

- An experiment conducted in a tall tower measured the “blueshift” change in frequency of a light pulse sent down the tower. The energy gained when traveling downward a distance $H$ is $mgH$. If $f$ is the energy frequency of light at the top and $f'$ is the frequency at the bottom, energy conservation gives

$$hf = hf' + mgH.$$  

The effective mass of light is $m = E / c^2 = hf / c^2$.

This yields the ratio of frequency shift to the frequency:

$$\frac{\Delta f}{f} = \frac{gH}{c^2}.$$  

Or in general:

$$\frac{\Delta f}{f} = -\frac{GM}{c^2} \left( \frac{1}{r_1} - \frac{1}{r_2} \right).$$

Using gamma rays, the frequency ratio was observed to be:

$$\frac{\Delta f}{f} \approx 10^{-15}.$$
Two airplanes took off (at different times) from Washington, D.C., where the U.S. Naval Observatory is located. The airplanes traveled east and west around Earth as it rotated. Atomic clocks on the airplanes were compared with similar clocks kept at the observatory to show that the moving clocks in the airplanes ran differently.
The time is changing in the moving frame, but the calculations must also take into account corrections due to general relativity (Einstein). Analysis shows that the special theory of relativity is verified within the experimental uncertainties.
Repeat Example 15-1 with a more accurate expression

18. \[ \frac{\Delta T}{T} = -\frac{GM}{c^2} \left( \frac{1}{r_1} - \frac{1}{r_2} \right) \]

We use \( r_2 = 6378 \text{ km} \) and \( r_1 = (6378 + 10) \text{ km} \).

\[ \frac{\Delta T}{T} = -\left( \frac{6.673 \times 10^{-11} \text{ m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2}}{2.998 \times 10^8 \text{ m/s}} \right)^2 \left( \frac{5.98 \times 10^{24} \text{ kg}}{6388 \times 10^3 \text{ m}} \right) \left( \frac{1}{\frac{6388 \times 10^3 \text{ m}}{6378 \times 10^3 \text{ m}}} \right) \]

\[ = 1.09 \times 10^{-12} \]
EXAMPLE 15.5

In the test of Hafele and Keating of flying atomic clocks around Earth (see Chapter 2), the gravitational redshift had to be considered. Calculate the effect and compare it with the special relativity time dilation effect. Assume the jet airplane travels 300 m/s and the circumference of Earth is about \(4 \times 10^7\) m.

**Strategy** The ratio \(\Delta f/f\) will be equal to the time difference ratio \(\Delta T/T\) measured by the clocks on Earth and on the jet airplanes. Because the airplane’s altitude is negligible in comparison to Earth’s radius, we use the simpler of the two equations for the gravitational redshift, Equation (15.14). We use the time dilation Equation (2.19) from Chapter 2 to determine the special relativity effect.

**Solution** Equation (15.14) gives us

\[
\frac{\Delta T}{T} = \frac{gH}{c^2}
\]

Measuring from Earth’s center, the position of the clock on the Earth’s surface is \(r_e\), and the position of the flying airplane is \(r_e + A\), where \(A\) is the altitude of the airplane, about 33,000 feet (10,000 m). The value of \(H\) is the difference in the position of the two clocks in Earth’s gravitational field \(g\), and \(H = A\). We neglect the change of \(g\) over this altitude, and \(\Delta T/T\) becomes

\[
\frac{\Delta T}{T} = \frac{(9.8 \text{ m/s}^2)(10,000 \text{ m})}{(3 \times 10^8 \text{ m/s})^2} = 1.09 \times 10^{-12}
\]

The eastward and westward airplane trips took about \(T = 45\) hours flying time. The difference in the two clocks due to the gravitational redshift is

\[
\Delta T = (1.09 \times 10^{-12})(45 \text{ h}) \left(\frac{3600 \text{ s}}{1 \text{ h}}\right)
\]

\[
= 177 \times 10^{-9} \text{ s} = 177 \text{ ns}
\]

A clock fixed on Earth will measure a flight time \(T_0\) of

\[
T_0 = \frac{4 \times 10^7 \text{ m}}{300 \text{ m/s}} = 1.33 \times 10^5 \text{ s}
\]
Find the relative frequency shift for light emitted from the sun, if the light is received at

a. the planet Mercury

b. the earth
Gravitational Time Dilation

• A very accurate experiment was done by comparing the frequency of an atomic clock flown on a Scout D rocket to an altitude of 10,000 km with the frequency of a similar clock on the ground. The measurement agreed with Einstein’s general relativity theory to within 0.02%.

• Since the frequency of the clock decreases near the Earth, a clock in a gravitational field runs more slowly according to the gravitational time dilation.
As light passes by a massive object, the path taken by the light is longer because of the spacetime curvature. The longer path causes a time delay for a light pulse traveling close to the sun. This effect was measured by sending a radar wave to Venus, where it was reflected back to Earth. The position of Venus had to be in the “superior conjunction” position on the other side of the sun from the Earth. The signal passed near the sun and experienced a time delay of about 200 microseconds. This was in excellent agreement with the general theory.
Perihelion Shift of Mercury

- The orbits of the planets are ellipses, and the point closest to the sun in a planetary orbit is called the perihelion. It has been known for hundreds of years that Mercury’s orbit processes about the sun. Accounting for the perturbations of the other planets left 43 seconds of arc per century that was previously unexplained by classical physics.

- The curvature of spacetime explained by general relativity accounted for the 43 seconds of arc shift in the orbit of Mercury.
15.4: Black Holes

- While a star is burning, the heat produced by the thermonuclear reactions pushes out the star’s matter and balances the force of gravity. When the star’s fuel is depleted, no heat is left to counteract the force of gravity, which becomes dominant. The star’s mass collapses into an incredibly dense ball that could warp spacetime enough to not allow light to escape. The point at the center is called a singularity.

- A collapsing star greater than 3 solar masses will distort spacetime in this way to create a black hole.
- Karl Schwarzschild determined the radius of a black hole now known as the Schwarzschild radius.

\[ r_s = \frac{2GM}{c^2} \]
Black holes

Escape velocity: initial kinetic energy = potential barrier of the gravitational field.

\[ \frac{mv^2}{2} = \frac{GMm}{r} \]

\[ r = \frac{2GM}{v^2} \]

\[ v = c \]

\[ r_S = \frac{2GM}{c^2} \quad \text{Schwarzschild radius} \]

\[ r_S = \frac{2GM}{c^2} = (1.5 \times 10^{-27} \text{ m/kg}) M \]
What is the Schwarzschild radius of
a. the moon?
b. Jupiter?
Black Hole Detection

• Since light can’t escape, they must be detected indirectly:
• Severe redshifting of light.
• *Hawking radiation* results from particle-antiparticle pairs created near the event horizon. One member slips into the singularity as the other escapes. Antiparticles that escape radiate as they combine with matter. Energy expended to pair production at the event horizon decreases the total mass-energy of the black hole.
• Hawking calculated the blackbody temperature of the black hole to be:

\[ T = \frac{\hbar c^3}{8\pi kGM} \]

• The power radiated is:

\[ P(T) = 4\pi\sigma r_s^2 \left( \frac{\hbar c^3}{8\pi kGM} \right)^4 \]

This result is used to detect a black hole by its Hawking radiation.

• Mass falling into a black hole would create a rotating *accretion disk*. Internal friction would create heat and emit x rays.
For a black hole with the mass of our moon find

a. The Schwarzschild radius
b. The effective temperature
c. The potential energy associated with the black hole being just above the earth’s surface

(a) \( r_s = \frac{2GM}{c^2} = \frac{2(6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2)(7.3 \times 10^{22} \text{ kg})}{(3.0 \times 10^8 \text{ m/s})^2} = 1.1 \times 10^{-4} \text{ m} = 0.11 \text{ mm} \)

(b) \( T = \frac{\hbar c^3}{8\pi kGM} = \frac{(1.05 \times 10^{-34} \text{ J} \cdot \text{s})(3.0 \times 10^8 \text{ m/s})^3}{8\pi (1.38 \times 10^{-23} \text{ J/K})(6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2)(7.3 \times 10^{22} \text{ kg})} = 1.7 \text{ K} \)

(c) \( U = -\frac{GMm}{R} = -\frac{(6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2)(7.3 \times 10^{22} \text{ kg})(6.0 \times 10^{24} \text{ kg})}{6.4 \times 10^6 \text{ m}} = -4.6 \times 10^{30} \text{ J} \)

This is a huge amount of gravitational energy.
Evidence of Black Holes

- Although a black hole has not yet been observed, there are several plausible candidates:
  - Cygnus X-1 is an x-ray emitter and part of a binary system in the Cygnus constellation. It is roughly 9 solar masses.
  - The galactic center of M87 is thought to be at least 3 billion solar masses.
  - NGC 4261 is thought to be a billion solar masses.
Active Galactic Nuclei (AGN) and Quasars

• **Active galactic nuclei** is a category of exotic objects that includes: luminous quasars, Seyfert galaxies, and blazars.

• Many believe the core of an AGN contains a supermassive black hole surrounded by an accretion disk. As matter spirals in the black hole, electromagnetic radiation and plasma jets spew outward from the poles.

• Blazars are AGN with jets spewing relativistic energies toward the Earth.
Gamma Ray Astrophysics

- Gamma-ray bursts (GRBs) are short flashes of electromagnetic radiation that are observed about once a day at unpredictable times from random directions.
- GRBs are absorbed in the atmosphere so they are observed by satellites.
- They last from a few milliseconds to several minutes.
- They were recently discovered to come from supernovae in distant galaxies.
- An interesting property of GRBs is the afterglow of lower energy photons including x rays, light and radio waves that last for weeks.
- The optical spectra of the GRBs is nearly identical to the jet of a supernova. Astronomers in 1997 were able to prove the bursts came from far away. The directional measurements proved that GRBs came from random directions as well.
15.3: Gravitational Waves

- When a charge accelerates, the electric field surrounding the charge redistributes itself. This change in the electric field produces an electromagnetic wave, which is easily detected. In much the same way, an accelerated mass should also produce gravitational waves.
- Gravitational waves carry energy and momentum, travel at the speed of light, and are characterized by frequency and wavelength.
- As gravitational waves pass through spacetime, they cause small ripples. The stretching and shrinking is on the order of 1 part in $10^{21}$ even due to a strong gravitational wave source.
- Due to their small magnitude, gravitational waves are difficult to detect. Large astronomical events create measurable spacetime waves such as the collapse of a neutron star, a black hole or the Big Bang.
- This effect has been likened to noticing a single grain of sand added to all the beaches of Long Island, New York.
Gravitational Wave Experiments (1 of 2)

• Taylor and Hulse discovered a binary system of two neutron stars that lose energy due to gravitational waves that agrees with the predictions of general relativity.

• LIGO is a large Michelson interferometer device that uses four test masses on two arms of the interferometer. The device is meant to detect changes in length of the arms due to a passing wave.

• NASA and the European Space Agency (ESA) were jointly developing a space-based probe called the Laser Interferometer Space Antenna (LISA), which was to measure fluctuations in its triangular shape.
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Gravitational Wave Experiments (2 of 2)

- The merger of the two neutron stars was also detected electromagnetically (as a short-period GRB) by the *Fermi Gamma-ray Space Telescope* less than 2 seconds after the gravitational wave detection.

Figure shows (Top panel) A schematic of the merger of two black holes. (Middle panel) The model of the gravitational wave signal detected from the merger. (Bottom panel) The separation distance (brown line) and the relative velocity of the black holes (green line) during the merger. *B. P. Abbott et al*, Physical Review Letters, 116, 061102 (2016).
GRAVITATIONAL WAVES DETECTED FROM COLLIDING NEUTRON STARS

Emmitt both gravitational waves and light
Cosmological Observations

- Stellar Evolution
- Galaxies and the Discovery of Dark Matter

By fitting a theoretical model of the composition of the universe to the combined set of cosmological observations, scientists have come up with the composition that we described above, ~68% dark energy, ~27% dark matter, ~5% normal matter. What is dark matter?
Stellar Evolution

• As the universe cooled, gravitational forces attracted the matter into gaseous clouds, which formed the basis of stars.

• This process continued as the interior temperature and density of these clouds increased.

• Nuclear fusion began when the temperature reached $10^7 \text{ K}$.

• Initially, fusion created helium from the hydrogen nuclei. Then further processes created carbon and heavier elements up to iron.
The Ultimate Fate of Stars

- The final stages of a star occur when the hydrogen fuel is exhausted and helium fuses. Heavier elements are then created until the process reaches the iron region.
- At this point the elements in the star have the highest binding energy per nucleon and the fusion reactions end.
- For $N$ nucleons each of mass $m$, the potential energy of a sphere of mass $Nm$ and radius $R$ is

$$P.E. = U_{\text{grav}} = -\frac{3}{5} \frac{G(Nm)^2}{R}$$

- The gravitational pressure is

$$P_{\text{grav}} = 0.32G \frac{(Nm)^2}{V^{4/3}}$$
Neutron Stars

• Matter is kept from total collapse by the outward electron pressure due to the Pauli exclusion principle. For massive stars, the gravity will force the electrons to interact with the protons:

\[ e^- + p \rightarrow n + \nu_e \]

• This result is called a neutron star from the abundance of neutrons. Similarly, the neutrons have an outward pressure:

\[ P_e = \frac{2f \pi^2}{3} \frac{\hbar^2}{2m_e} \left( \frac{N_e}{V} \right)^{5/3} \]

• Balancing these pressures yields the volume of a neutron star:

\[ V^{1/3} = \frac{6.5\hbar^2}{N^{1/3} m^3 G} \]
Novae and Supernovae

- **Novae** and **supernovae** are stars that brighten and then fade.
- Type I supernovae have no hydrogen spectral lines and type II do.
- Type Ia supernovae are the brightest and are thought to be collapsing white dwarf stars.
- Cataclysmic explosions in supernovae provide the temperature and pressure to produce heavier elements such as uranium.
- The Crab supernova occurred in 1054 and was recorded by the Chinese and Japanese. It was bright enough to see during the daytime.
- Other supernovae occurred in 1572, 1604 and 1987.
Supernova Explosion

SN 1987A Supernova

As most of the heavier elements fused into iron, the iron nuclei became so hot that they spewed out helium nuclei. The temperature and density were large enough to radiate neutrinos.

- The gravitational force was strong enough to form a neutron star.
- The implosion rebounded from the repulsive strong nuclear force in the core and created a dense shockwave. The shockwave radiated neutrinos out from the star.
- These neutrinos were detected in Japan and the U.S. three hours before the light reached the Earth.
- The neutrino observations were consistent with the supernova predictions.
Galaxies

- Collections of stars, gas, and dust are called **galaxies**.
- The number of galaxies is extremely large, at least 100 billion.
- Our own galaxy is the **Milky Way**, and it is believed to be composed of about 200 billion stars.
- Andromeda is the closest galaxy within a million light-years.
Galaxies

As the gas and stars in a spiral galaxy orbit, the spectral lines can be seen to be redshifted when material is moving away from us and blueshifted when moving toward us.
Dark Energy

One explanation for dark energy is that it is a property of space. Albert Einstein was the first person to realize that empty space is not nothing. Space has amazing properties, many of which are just beginning to be understood. The first property that Einstein discovered is that it is possible for more space to come into existence. Then one version of Einstein’s gravity theory, the version that contains a cosmological constant, makes a second prediction: “empty space” can possess its own energy. Because this energy is a property of space itself, it would not be diluted as space expands. As more space comes into existence, more of this energy-of-space would appear. As a result, this form of energy would cause the universe to expand faster and faster.

Unfortunately, no one understands why the cosmological constant should even be there, much less why it would have exactly the right value to cause the observed acceleration of the universe.

Universe Dark Energy-1 Expanding Universe

This diagram reveals changes in the rate of expansion since the universe's birth 15 billion years ago. The more shallow the curve, the faster the rate of expansion. The curve changes noticeably about 7.5 billion years ago, when objects in the universe began flying apart as a faster rate. Astronomers theorize that the faster expansion rate is due to a mysterious, dark force that is pulling galaxies apart.

Credit: NASA/STSci/Ann Feild
1.4: Conservation Laws and Fundamental Forces

• Recall the fundamental conservation laws:
  • Conservation of energy
  • Conservation of linear momentum
  • Conservation of angular momentum
  • Conservation of electric charge

• Later we will establish the conservation of mass as part of the conservation of energy
Total Energy and Rest Energy, Mass-energy Equivalence

We rewrite the energy equation in the form

$$\gamma mc^2 = \frac{mc^2}{\sqrt{1-u^2/c^2}} = K + mc^2$$  \hspace{1cm} (2.63)

The term $mc^2$ is called the rest energy and is denoted by $E_0$.

$$E_0 = mc^2$$  \hspace{1cm} (2.64)

This leaves the sum of the kinetic energy and rest energy to be interpreted as the total energy of the particle. The total energy is denoted by

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1-u^2/c^2}} = \frac{E_0}{\sqrt{1-u^2/c^2}} = K + E_0$$  \hspace{1cm} (2.65)
The Equivalence of Mass and Energy

- By virtue of the relation for the rest mass of a particle:

  \[ E_0 = mc^2 \]

- we see that there is an equivalence of mass and energy in the sense that “mass and energy are interchangeable”

- Thus the terms mass-energy and energy are sometimes used interchangeably.