The Relativistic Doppler Effect (con’t)

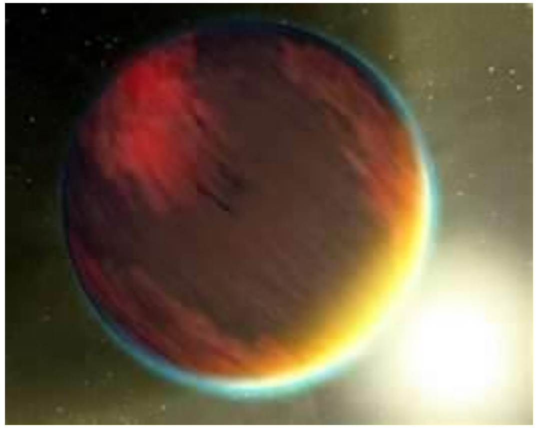
Equations (2.32) and (2.33) can be combined into one equation if we agree to use a + sign for β ($+v/c$) when the source and receiver are approaching each other and a – sign for β ($-v/c$) when they are receding. The final equation becomes

$$f = \frac{\sqrt{1+\beta}}{\sqrt{1-\beta}} f_0 \qquad \text{Relativistic Doppler effect} \quad (2.34)$$

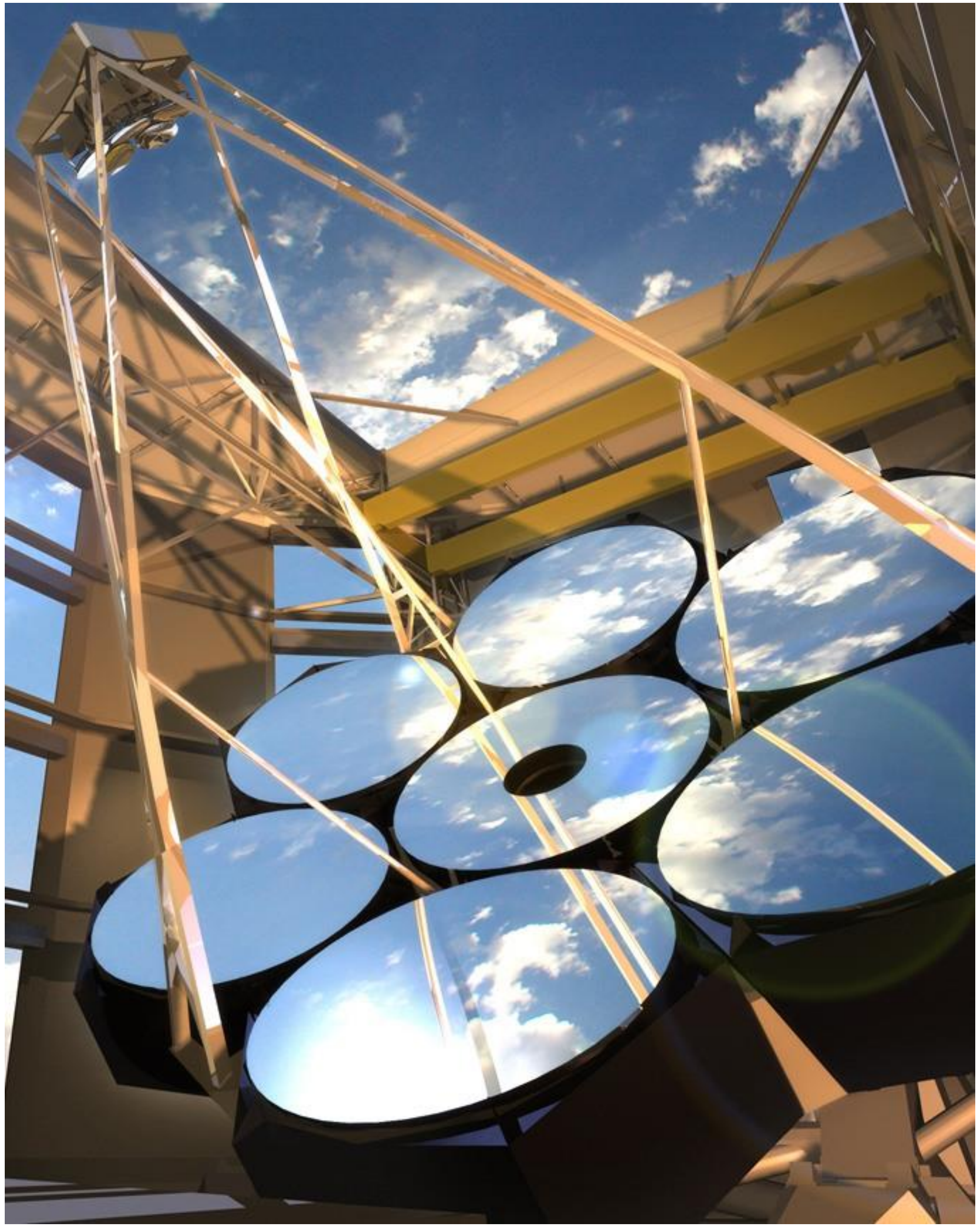
Application of the Doppler shift in Astronomy

Detecting Extrasolar Planets

- Great effort is made to discover earth-like planets in distant solar systems
- Various techniques to detect Exoplanets via:
  - the additional redshift caused by the star's motion around a common center of gravity
  - the induced change in position of its star
  - the dimming of the star's brightness during the transition of a planet
  - the induced change of a another planet's orbit



Habitable Earth-like exoplanets?



Touching Modern Physics in 2020-25

Giant Magellan Telescope

is a ground-based extremely large telescope under construction. It will consist of seven 8.4 m (27.6 ft) diameter primary segments,[1] that will observe optical and near infrared (320–25000 nm[2]) light, with the resolving power of a 24.5 m (80.4 ft) primary mirror and collecting area equivalent to a 22.0 m (72.2 ft) one,[3] which is about 368 square meters.[4] The telescope is expected to have a resolving power 10 times greater than the Hubble Space Telescope.

StoRy of GoldiLocks & the 3 bears

Once upon a time, there was a little girl named Goldilocks. She went for a walk in the forest. Pretty soon, she came upon a house. She knocked, and when no one answered, she walked right in…

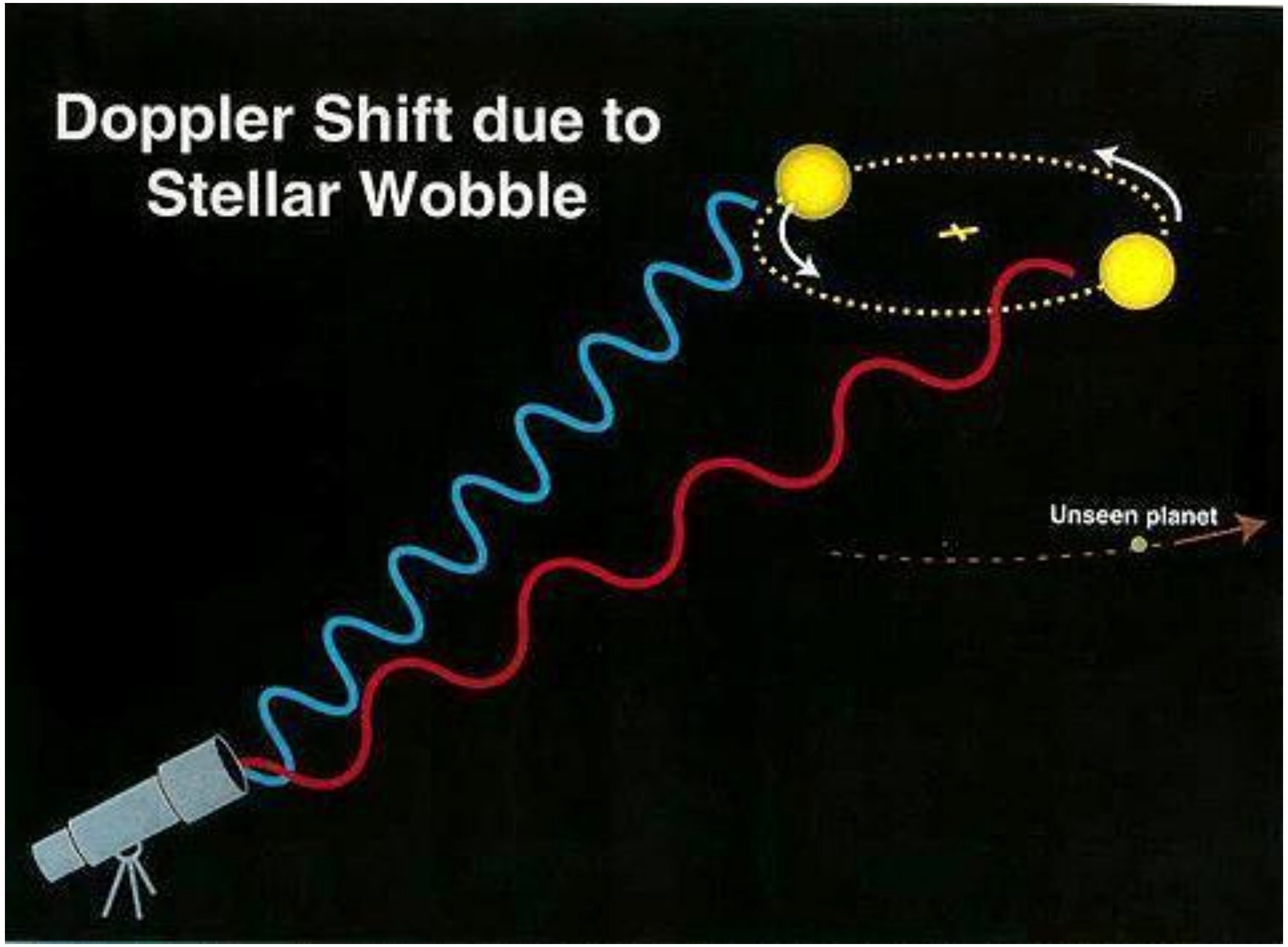
At the table in the kitchen, there were three bowls of porridge.

- "This porridge is **too hot**!"
- "This porridge is **too cold**!"
- "This porridge is **just right**!"

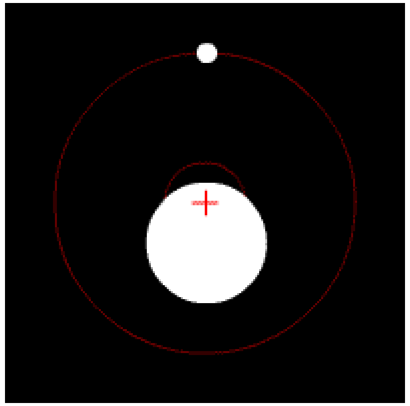
She also tried out each of the three chairs and three beds.

- **Too big**, **Too small**, **Too hard**, **Too soft**, and **Just Right**

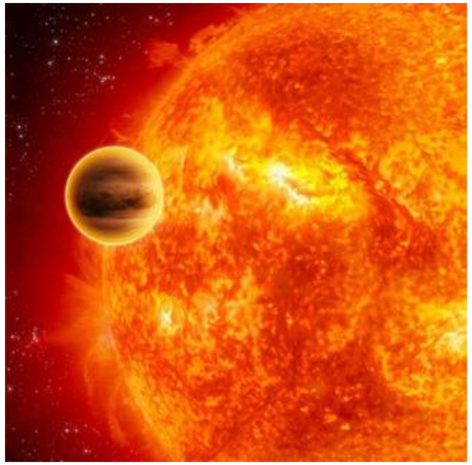
RV search for exo - Planets



Radial Velocity Variations induced by exo-planets

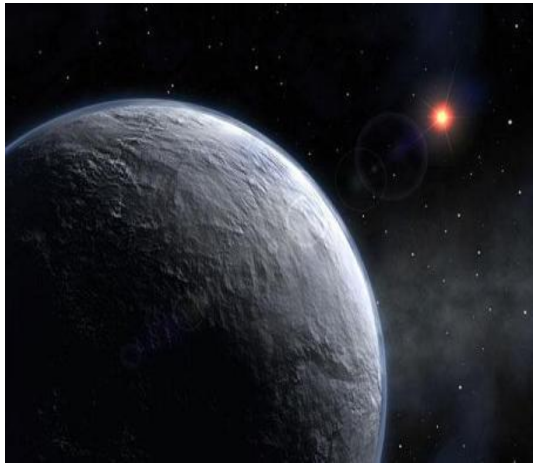


Detecting Extra Solar Planets



- As of September 1 2022 there are 5,157 exoplanets discovered
- smallest planet detected so far: 5-Earth-mass

- corresponding precision: **60 cm/s** required
- precision for an Earth-mass object in an Earth-like orbit around a Sun-like star: **5 cm/s (50 kHz)**
- long term stability **over years**



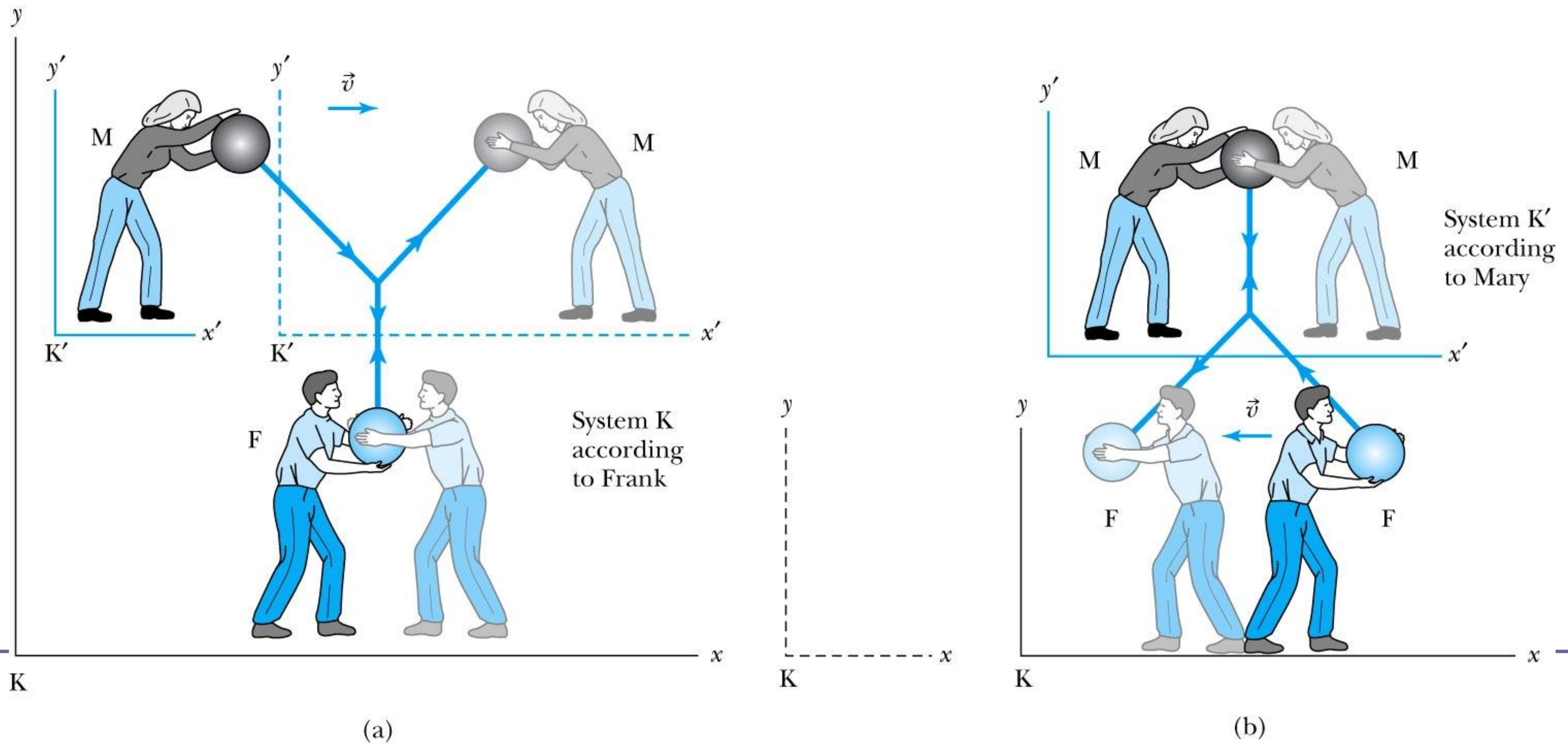
2.11: Relativistic Momentum

Because physicists believe that the conservation of momentum is fundamental, we begin by considering collisions where there do not exist external forces and

$$d\mathbf{P}/dt = \mathbf{F}_{\mathbf{ext}} = \mathbf{0}$$

Relativistic Momentum

Frank (fixed or stationary system) is at rest in system K holding a ball of mass *m*. Mary (moving system) holds a similar ball in system K that is moving in the *x* direction with velocity *v* with respect to system K.



Relativistic Momentum

- If we use the definition of momentum, the momentum of the ball thrown by Frank is entirely in the *y* direction:

$$p_{Fy} = mu_0$$

The change of momentum as observed by Frank is

$$\Delta p_F = \Delta p_{Fy} = -2mu_0$$

According to Mary (the **M**oving frame)

- Mary measures the initial velocity of her own ball to be $u'_{Mx} = 0$ and $u'_{My} = -u_0$.

  In order to determine the velocity of Mary's ball as measured by Frank we use the velocity transformation equations:

$$u_{Mx} = v$$

$$u_{My} = -u_0\sqrt{1 - v^2/c^2}$$

(we used velocity summation formula

$$u'_y = \frac{u_y}{\gamma\left[1 - (v/c^2)u_x\right]}$$

with $u_x=0$)

Relativistic Momentum

Before the collision, the momentum of Mary's ball as measured by Frank (the **F**ixed frame) becomes

Before $p_{Mx} = mv$

Before $p_{My} = -mu_0\sqrt{1 - v^2/c^2}$ (2.42)

For a perfectly elastic collision, the momentum after the collision is

After $p_{Mx} = mv$

After $p_{My} = +mu_0\sqrt{1 - v^2/c^2}$ (2.43)

The change in momentum of Mary's ball according to Frank is

$$\Delta p_M = \Delta p_{My} = 2mu_0\sqrt{1 - v^2/c^2} \quad (2.44)$$

Relativistic Momentum (con’t)

- The conservation of linear momentum requires the total change in momentum of the collision, $\Delta p_F + \Delta p_M$, to be zero. The addition of Equations (2.40) and (2.44) clearly does not give zero.

$$\Delta p_F = \Delta p_{Fy} = -2mu_0 \qquad \Delta p_M = \Delta p_{My} = 2mu_0\sqrt{1 - v^2/c^2}$$

- *Linear momentum is not conserved if we use the conventions for momentum from classical physics even if we use the velocity transformation equations from the special theory of relativity.*

- There is no problem with the *x* direction, but there is a problem with the *y* direction along the direction the ball is thrown in each system.

Relativistic Momentum

- Rather than abandon the conservation of linear momentum, let us look for a modification of the definition of linear momentum that preserves both it and Newton's second law.
- To do so requires reexamining mass to conclude that:

$$\vec{p} = m\frac{d\vec{r}}{dt}\gamma$$

$$\boxed{p = \gamma m\vec{u} \qquad \text{Relativistic momentum (2.48)}}$$

$$\gamma = \frac{1}{\sqrt{1-u^2/c^2}}$$

With modified (relativistic) momentum (Example 2.9)

$$\Delta p_{Fy} = -\frac{2mu_0}{\sqrt{1-u_0{}^2/c^2}}$$

$$\Delta p_{My} = \frac{2m_0u_0\sqrt{1-v^2/c^2}}{\sqrt{1-[v^2+u_0{}^2(1-v^2)/c^2]/c^2}} =$$

$$\frac{2m_0u_0\sqrt{1-v^2/c^2}}{\sqrt{(1-v^2/c^2)(1-u_0{}^2/c^2)}} = \frac{2m_0u_0}{\sqrt{(1-u_0{}^2/c^2)}}$$

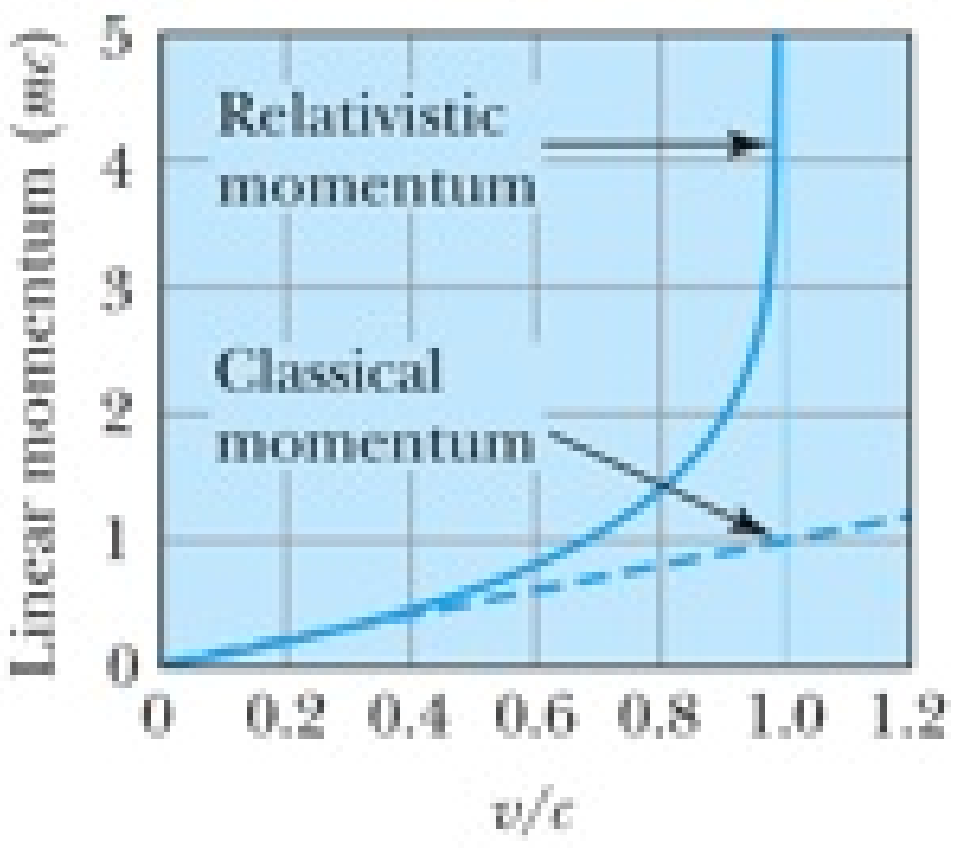
Now $\Delta p_F + \Delta p_M = 0$
and momentum conserved!

Relativistic Momentum: two points of view

- physicists like to refer to the mass in Equation (2.48) as the *rest mass* m_0 and call the term $m = \gamma m_0$ the *relativistic mass*. In this manner the classical form of momentum, p=*mu*, is retained. The mass is then imagined to increase at high speeds.

- physicists prefer to keep the concept of mass as an invariant, intrinsic property of an object. We adopt this latter approach and will use the term *mass* exclusively to mean *rest mass*.

Behavior of relativistic momentum and classical momentum for v/c->1



2.11
#60

A particle initially has a speed of $0.5c$. At what speed does its momentum increase by (a) 1%, (b) 10%, (c) 100%?

(a)

Initial momentum $P_0 = \gamma_0 m u_0 = \frac{1}{\sqrt{1-(0.5)^2}} \cdot m \cdot (0.5c)$

$= (0.57735c) \cdot m$

1% increase

$P = P_0(1.01) \Rightarrow \gamma_1 \cancel{m} u_1 = (0.57735c)\cancel{m} \cdot (1.01)$

$\gamma_1 u_1 \doteq (0.58312c)$

$$\frac{u_1}{\sqrt{1-\frac{u_1^2}{c^2}}} = (0.58c) \Rightarrow \frac{u_1^2}{\left(1-\frac{u_1^2}{c^2}\right)} = (0.58c)^2 \Rightarrow$$

$(0.58)^2 c^2$

$$\Rightarrow u_1^2 = (0.58)^2 c^2 \left(1-\frac{u_1^2}{c^2}\right) \Rightarrow u_1^2 = (0.58)^2 c^2 - \frac{(0.58)^2 \cancel{c^2} u_1^2}{\cancel{c^2}} \Rightarrow$$

$$\Rightarrow u_1^2 + (0.58)^2 u_1^2 = (0.58)^2 c^2 \Rightarrow u_1^2\left(1+(0.58)^2\right) = (0.58)^2 c^2 \Rightarrow$$

$$\Rightarrow u_1^2 = \frac{(0.58)^2 c^2}{1+(0.58)^2} \Rightarrow u_1 = \frac{(0.58)\,c}{\sqrt{1+(0.58)^2}} \Rightarrow \boxed{u_1 = 0.5037c}$$

Relativistic Kinetic Energy

In classical mechanics; $k = \frac{1}{2}mv^2$. What is the kinetic energy ***k*** in special relativity?

$$F = \frac{dP}{dt} = \frac{d}{dt}(\gamma m_o u) = \frac{d}{dt}\frac{m_o u}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$w_{12} = \int_1^2 \vec{F}\,d\vec{s} = k_2 - k_1$$

Assume; k_1 = 0, that is particle starts from rest and $ds = u\,dt$

$$w = K = \int \frac{d}{dt}(\gamma m_o u)\,\vec{u}dt = m_o \int dt\frac{d}{dt}(\gamma u)\vec{u} = m_o \int u\,d(\gamma\,u)$$

By using integration by parts;

$$\int x\,dy = xy - \int y\,dx \qquad \therefore x = u \;\; and \;\; y = \gamma u$$

$$\int u\,d(\gamma u) = \gamma u^2 - \int \gamma u\,du$$

$$k = m_o \int_0^u u\,d(\gamma\,u) = m_o\gamma u^2 - m_o \int \gamma u\,du$$

$$k = m_o\gamma u^2 - m_o \int \frac{u}{\sqrt{1-\frac{u^2}{C^2}}}du$$

Integral tables give below equation;

$$K = m_o\gamma u^2 \; - m_o\, c^2\sqrt{1-\frac{u^2}{c^2}}\,\Bigg|\begin{matrix} u \\ 0 \end{matrix} = m_o\gamma u^2 + m_o c^2\sqrt{1-\frac{u^2}{c^2}} - m_o c^2$$

$$K = \frac{m_o u^2 + m_o c^2(1-\frac{u^2}{c^2})}{\sqrt{1-\frac{u^2}{c^2}}} - m_o c^2 = m_o\gamma c^2 - m_o c^2$$

$$\boxed{K = m_o c^2(\gamma - 1)}$$

This equation reduces to the classical form of $\frac{1}{2}mv^2$ for v<<c.

Relativistic Kinetic Energy

Equation (2.58) does not seem to resemble the classical result for kinetic energy, $K = \frac{1}{2}mu^2$. However, if it is correct, we expect it to reduce to the classical result for low speeds. Let's see if it does. For speeds $u \ll c$, we expand γ in a binomial series as follows:

$$K = mc^2\left(1 - \frac{u^2}{c^2}\right)^{-1/2} - mc^2$$

$$= mc^2\left(1 + \frac{1}{2}\frac{u^2}{c^2} + \dots\right) - mc^2$$

where we have neglected all terms of power $(u/c)^4$ and greater, because $u \ll c$. This gives the following equation for the relativistic kinetic energy at low speeds:

$$K = mc^2 + \frac{1}{2}mu^2 - mc^2 = \frac{1}{2}mu^2$$

which is the expected classical result.

Total Energy and Rest Energy, Mass-energy Equivalence

We rewrite the energy equation in the form

$$\gamma mc^2 = \frac{mc^2}{\sqrt{1-u^2/c^2}} = K + mc^2 \qquad (2.63)$$

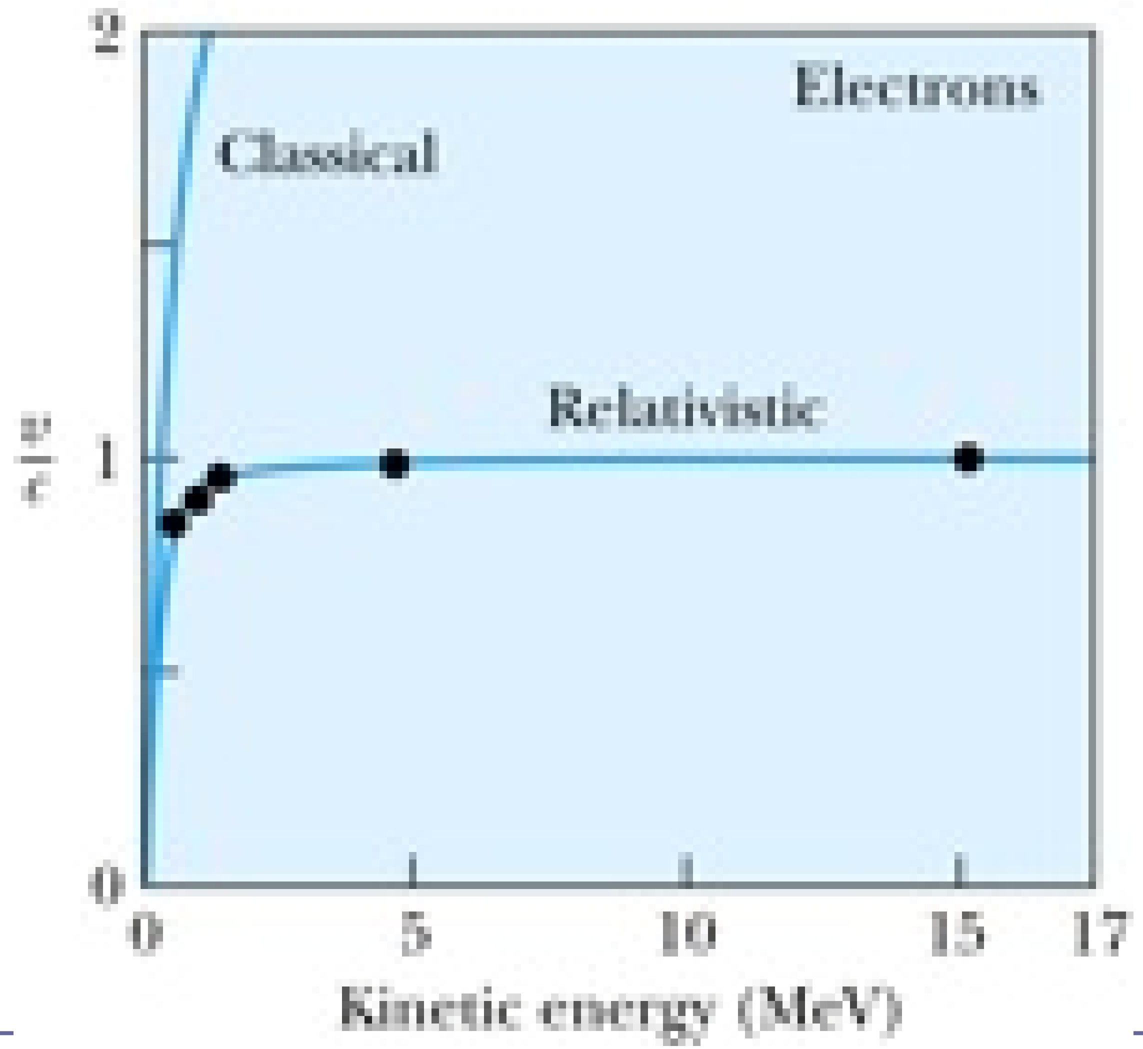
The term mc^2 is called the rest energy and is denoted by E_0.

$$E_0 = mc^2 \qquad (2.64)$$

This leaves the sum of the kinetic energy and rest energy to be interpreted as the total energy of the particle. The total energy is denoted by E and is given by

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1-u^2/c^2}} = \frac{E_0}{\sqrt{1-u^2/c^2}} = K + E_0 \qquad (2.65)$$

Kinetic Energy-Velocity (Relativistic and Classical)



The Equivalence of Mass and Energy

- By virtue of the relation for the rest mass of a particle:

$$\boxed{E_0 = mc^2}$$

- we see that there is an *equivalence* of mass and energy in the sense that "mass and energy are interchangeable"

- Thus the terms *mass-energy* and *energy* are sometimes used interchangeably.

2.12#70

What is the speed of a proton when its kinetic energy is equal to twice its rest energy?

$E = K + E_0 = 2E_0 + E_0 = 3E_0 = \gamma E_0 \rightarrow \gamma = 3$

then $\beta = \sqrt{1 - \frac{1}{\gamma^2}} = \sqrt{1 - \frac{1}{3^2}} = 0.943 = \frac{v}{c}$

$$\boxed{v = 0.943c}$$

Problem 2.12 #75

How much mass-energy (in joules) is contained in a peanut weighing 0.1 ounce? How much mass-energy do you gain by eating 10 ounces of peanuts? Compare this with the food energy content of peanuts, about 100 kcal per ounce.

1. Converting 0.1 ounce = 2.835×10^{-3} kg.

$$E = mc^2 = \left(2.835 \times 10^{-3}\ \text{kg}\right)\left(3.00 \times 10^{8}\ \text{m/s}\right)^2 = 2.55 \times 10^{14}\ \text{J}.$$

Eating 10 ounces results in a factor of 100 greater mass-energy increase, or 2.55×10^{16} J. This is a small increase compared with your original mass-energy, but it will tend to increase your weight

Relationship of Energy and Momentum

$$p = \gamma m u = \frac{mu}{\sqrt{1 - u^2/c^2}}$$

We square this result, multiply by c^2, and rearrange the result.

$$p^2c^2 = \gamma^2 m^2 u^2 c^2$$

$$= \gamma^2 m^2 c^4 \left(\frac{u^2}{c^2}\right) = \gamma^2 m^2 c^4 \beta^2$$

We use the equation for γ to express β^2 and find

Expressing β through γ

$$\gamma^2 = 1/(1-\beta^2) \quad \beta^2 = (\gamma^2 - 1)/\gamma^2 = 1 - 1/\gamma^2$$

$$p^2c^2 = \gamma^2 m^2 c^4 \left(1 - \frac{1}{\gamma^2}\right)$$

$$= \gamma^2 m^2 c^4 - m^2 c^4$$

Energy and Momentum

The first term on the right-hand side is just E^2, and the second term is E_0^2. The last equation becomes

$$p^2c^2 = E^2 - E_0^2$$

We rearrange this last equation to find the result we are seeking, a relation between energy and momentum.

$$E^2 = p^2c^2 + E_0^2 \qquad (2.70)$$

or

$$E^2 = p^2c^2 + m^2c^4 \qquad (2.71)$$

Equation (2.70) is a useful result to relate the total energy of a particle with its momentum. The quantities ($E^2 - p^2c^2$) and m are invariant quantities. Note that when a particle's velocity is zero and it has no momentum, Equation (2.70) correctly gives E_0 as the particle's total energy.

Useful formulas

$\beta = pc / E$ from $p = \gamma m u$ and $E = \gamma m c^2$

$$p = \frac{1}{c}(E^2 - E_0{}^2)^{1/2} = \frac{E}{c}\left[1 - \left(\frac{E_0}{E}\right)^{1/2}\right]$$

$$\beta = \left[1 - \left(\frac{E_0}{E}\right)^2\right]^{1/2} = \left(1 - \frac{1}{\gamma^2}\right)^{1/2}$$

Chapter 2 - 82

What is the kinetic energy of (a) an electron having a momentum of 40 GeV/c ? (b) a proton having a momentum of 40 GeV/c

(a) $$E^2 = p^2c^2 + E_0^2 \qquad E = \sqrt{p^2c^2 + E_0^2} = \sqrt{\left(\frac{40\ GeV}{c}\right)^2 c^2 + (511\ keV)^2}$$

$$= \sqrt{(40\ GeV)^2 + (511\ keV)^2} \simeq 40.0\ GeV$$

$$K = E - E_0 = 40.0\ GeV$$

(b) $$E = \sqrt{p^2c^2 + E_0^2} = \sqrt{(40\ GeV)^2 + (0.938\ GeV)^2} = 40.011\ GeV$$

$$K = E - E_0 = (40.011 - 0.938)\ GeV$$

$$\boxed{K = 39.07\ GeV}$$

94. An electron has a total energy that is 200 times its rest energy. Determine (a) its kinetic energy (b) its speed (c) its momentum

(a)

$$\left.\begin{aligned} K &= E - E_0 \\ K &= \underset{\substack{\uparrow \\ \text{total} \\ \text{energy}}}{\gamma m_0 c^2} - \underset{\substack{\uparrow \\ \text{rest} \\ \text{energy}}}{m_0 c^2} \end{aligned}\right\} K = (200-1)E_0 = 199E_0 = 199\,(511\,\text{keV}) = \boxed{102\ \text{MeV}}$$

(b)

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}} = 0.9999875 \qquad \boxed{v = 0.9999875\,c}$$

(c)

$$p^2c^2 = E^2 - E_0^2 \qquad E^2 = p^2c^2 + E_0^2 \qquad p = \frac{\sqrt{E^2 - E_0^2}}{c}$$

$$p = \frac{\sqrt{(200 \times 511\,\text{keV})^2 - (511\,\text{keV})^2}}{c} = \boxed{102\ \frac{\text{MeV}}{c}}$$

#2.95 A proton moves with a speed of 0.9c. (a) find the speed of an electron that has (a) the same momentum as the proton, and (b) the same kinetic energy.

(a) For the proton $p = \gamma m u = \frac{1}{\sqrt{1-0.9^2}} \times 938\,MeV/c^2 \times 0.9c = 1940\,\frac{MeV}{c}$

For the electron $E^2 = p^2c^2 + E_0^2 \Longrightarrow E = \sqrt{p^2c^2 + E_0^2} = \sqrt{(1940\,MeV)^2 + (0.511\,MeV)^2}$

$= 1940\,MeV$

$E = \gamma mc^2 = \gamma E_0 \qquad \gamma = \frac{E}{E_0} = \frac{1940\,MeV}{0.511\,MeV} = 3797$

$\beta = \sqrt{1 - \frac{1}{\gamma^2}} = \sqrt{1 - \frac{1}{3797^2}} = \sqrt{1 - 6.94\times10^{-8}} \approx 1 - \frac{1}{2}6.94\times10^{-8}$

$v = (1 - 3.97\times10^{-8})c$

(b) For the proton

$K = (\gamma - 1)E_0 = \left(\frac{1}{\sqrt{1-0.9^2}} - 1\right) 938\,MeV = 1214\,MeV$

For the electron

$\gamma E_0 = K + E_0 \qquad \gamma = \frac{1214\,MeV + 0.511\,MeV}{0.511\,MeV} = 2377$

$\beta = \sqrt{1 - \frac{1}{\gamma^2}} = \sqrt{1 - \frac{1}{2377^2}} = \sqrt{1 - 1.77\times10^{-7}} \approx 1 - 8.85\times10^{-8} \qquad v = (1 - 8.85\times10^{-8})c$

$\boxed{v \approx c}$

From chapter2 Quiz

- When relating the linear momentum and total energy of an object with speed v = 0.8c, which of the following changes would increase the energy by the greatest amount? Assume that it is possible to change the mass of the object.
- increase the momentum of the object to 2p while keeping the mass constant.
- increase the mass of the object to 2m while keeping the momentum constant.
- imagine the speed of light to change to 82% of its present value (the object is still at initial speed v).
- increase the momentum of the object to 2p while keeping the velocity constant.
- increase the speed of the object to 0.95c.

Massless particles have a speed equal to the speed of light c

- Recall that a photon has “zero” rest mass and that equation 2.70, from the last slide, reduces to: E = pc and we may conclude that:
- Thus the velocity, u, of a massless particle must be c since, as 0, and it follows that: u = c.

$$m \rightarrow \qquad \gamma \rightarrow ¥$$

$$E = \gamma mc^2 \ \ and \ \ pc = \gamma muc => \gamma mc^2 = \gamma muc$$

2.13: Computations in Modern Physics

- We were taught in introductory physics that the international system of units is preferable when doing calculations in science and engineering (“everyday” scales).
- In modern physics a somewhat different set of units is often used, which is more convenient for problems considered in modern physics.
- The smallness of quantities often used in modern physics suggests the need for some new units more practical for smaller scales .

Units of Work and Energy

- Recall that the work done in accelerating a charge through a potential difference is given by $W = qV$.
- For a proton, with the charge $e = 1.602 \times 10^{-19}$ C being accelerated across a potential difference of 1 V, the work done on the particle is

$$W = (1.602 \times 10^{-19}\,\mathrm{C})(1\,\mathrm{V}) = 1.602 \times 10^{-19}\,\mathrm{J}$$

The Electron Volt (eV)

- The work done to accelerate the proton across a potential difference of 1 V could also be written as

$$W = (1\ e)(1\ \text{V}) = 1\ \text{eV}$$

- Thus eV, pronounced "electron volt," is also a unit of energy. It is related to the SI (*Système International*) unit joule by the 2 previous equations.

$$1\ \text{eV} = 1.602 \times 10^{-19}\ \text{J}$$

Other Units

1) Rest energy of a particle:
   Example: E_0 (proton)

$$E_0(\text{proton}) = (1.67\times10^{-27}\ \text{kg})(3\times10^{8}\ \text{m/s})^2 = 1.50\times10^{-10}\ \text{J}$$

$$= 1.50\times10^{-10}\ \text{J}\ \frac{1\ \text{eV}}{1.602\times10^{-19}\ \text{J}} = 9.38\times10^{8}\ \text{eV}$$

2) **Atomic mass unit** (amu):

   Example: carbon-12

$$\text{Mass } (^{12}\text{C atom}) = \frac{12\ \text{g/mol}}{6.02\times10^{23}\ \text{atoms/mol}}$$

$$= 1.99\times10^{-23}\ \text{g/atom}$$

$$\text{Mass } (^{12}\text{C atom}) = 1.99\times10^{-26}\ \text{kg} = 12\ \text{u/atom}$$

1 u=1.66x10-27 kg=931.5 MeV/c^2

Two high energy protons hit each other headon

Solution (a) We use $K = 2.00$ GeV and the proton rest energy, 938 MeV, to find the total energy from Equation (2.65),

$$E = K + E_0 = 2.00 \text{ GeV} + 938 \text{ MeV} = 2.938 \text{ GeV}$$

The momentum is determined from Equation (2.70).

$$p^2c^2 = E^2 - E_0^2 = (2.938 \text{ GeV})^2 - (0.938 \text{ GeV})^2$$
$$= 7.75 \text{ GeV}^2$$

The momentum is calculated to be

$$p = \sqrt{7.75(\text{GeV}/c)^2} = 2.78 \text{ GeV}/c$$

Notice how naturally the unit of GeV/c arises in our calculation.

In order to find β we first find the relativistic factor γ. There are several ways to determine γ; one is to compare the rest energy with the total energy. From Equation (2.65) we have

$$E = \gamma E_0 = \frac{E_0}{\sqrt{1 - u^2/c^2}}$$

$$\gamma = \frac{E}{E_0} = \frac{2.938 \text{ GeV}}{0.938 \text{ GeV}} = 3.13$$

We use Equation (2.62) to determine β.

$$\beta = \sqrt{\frac{\gamma^2 - 1}{\gamma^2}} = \sqrt{\frac{3.13^2 - 1}{3.13^2}} = 0.948$$

The speed of a 2.00-GeV proton is $0.95c$ or 2.8×10^8 m/s.

(b) When the two protons collide head-on, the situation is similar to the case when the two blocks of wood collided head-on with one important exception. The time for the two protons to interact is less than 10^{-20} s. If the two protons did momentarily stop at rest, then the two-proton system would have its mass increased by an amount given by Equation (2.68), $2K/c^2$ or 4.00 GeV/c^2. The result would be a highly excited system. In fact, the collision between the protons happens very quickly, and there are several possible outcomes. The two protons may either remain or disappear, and new additional particles may be created. Two of the possibilities are

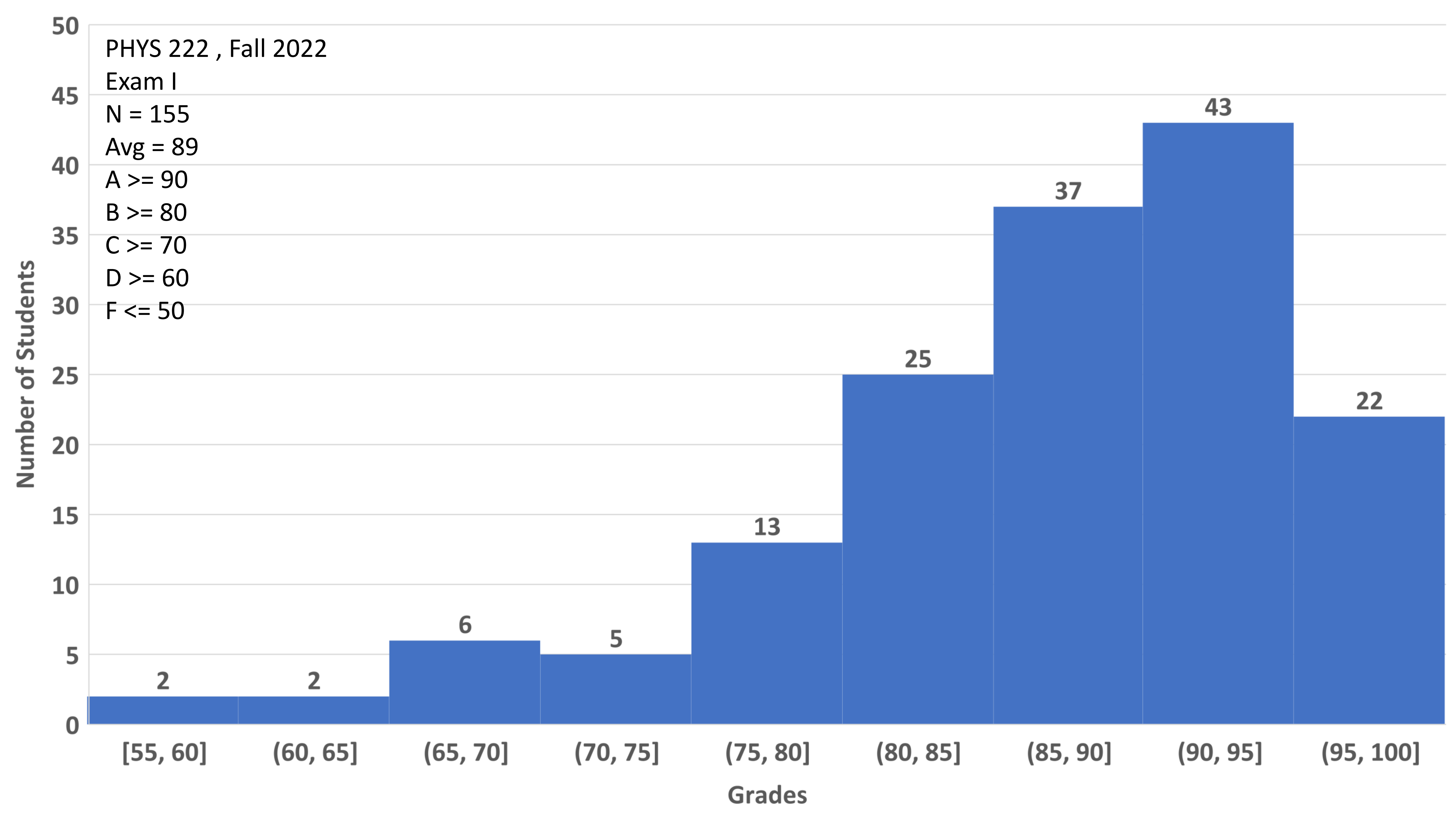
$$p + p \rightarrow p + p + p + \bar{p} \tag{2.78}$$

$$p + p \rightarrow \pi^+ + d \tag{2.79}$$

where the symbols are p (proton), $\bar{p}$ (antiproton), π (pion), and d (deuteron). We will learn more about the possibilities later when we study nuclear and particle physics. Whatever happens must be consistent with the conservation laws of charge, energy, and momentum, as well as with other conservation laws to be learned. Such experiments are routinely done in particle physics. In the analysis of these experiments, the equivalence of mass and energy is taken for granted.

Electromagnetism and Relativity

- Einstein was convinced that magnetic fields appeared as electric fields observed in another inertial frame. That conclusion is the key to electromagnetism and relativity.
- Einstein's belief that *Maxwell's equations describe electromagnetism in any inertial frame* was the key that led Einstein to the Lorentz transformations.
- Maxwell's assertion that all electromagnetic waves travel at the speed of light and Einstein's postulate that the speed of light is invariant in all inertial frames seem intimately connected.


PHYS 222 , Fall 2022
Exam I
N = 155
Avg = 89
A >= 90
B >= 80
C >= 70
D >= 60
F <= 50
Number of Students
50
45
40
35
30
25
20
15
10
5
0
2
2
6
5
13
25
37
43
22
[55, 60]
(60, 65]
(65, 70]
(70, 75]
(75, 80]
(80, 85]
(85, 90]
(90, 95]
(95, 100]
Grades

Thank you for your attention!