Ray tracing in an optical system

1. Ray Tracing in an Optical System

Ray Tracing in an Optical system (paraxial rays)

Since a light beam does not spread a lot we consider it as a ray

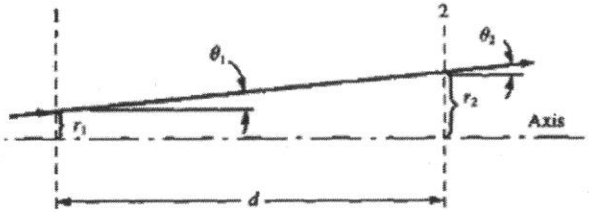


FIGURE 2.1. Ray in a homogeneous dielectric of length d.

ray $\triangleq$ path of the center of a very slowly diverging electromagnetic beam

$$\boxed{\text{paraxial} \triangleq \operatorname{tg}\theta \sim \sin\theta \sim \theta}$$

two lengths of space in fig 2.2

one length of space:

$$r_2 = 1 r_1 + d r_1'$$
$$r_2' = 0 r_1 + 1 r_1'$$

$$\begin{bmatrix} r_2 \\ r_2' \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_1' \end{bmatrix}$$

$$\boxed{T = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}}$$ free space length

general

$$\boxed{\begin{bmatrix} r_{out} \\ r_{out}' \end{bmatrix} = \underbrace{\begin{bmatrix} A & B \\ C & D \end{bmatrix}}_{T} \begin{bmatrix} r_{in} \\ r_{in}' \end{bmatrix}}$$

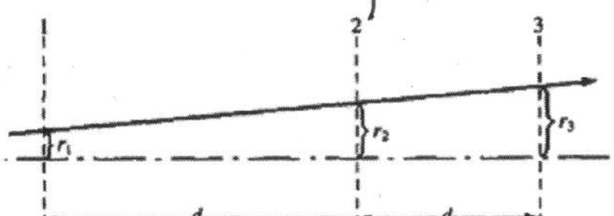


FIGURE 2.2. Example of two lengths of space.

matrix for two lengths of space:

$$\begin{bmatrix} r_3 \\ r_3' \end{bmatrix} = \begin{bmatrix} 1 & d_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_2 \\ r_2' \end{bmatrix}; \quad \begin{bmatrix} r_2 \\ r_2' \end{bmatrix} = \begin{bmatrix} 1 & d_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_1' \end{bmatrix}$$

$$\begin{bmatrix} r_3 \\ r_3' \end{bmatrix} = \begin{bmatrix} 1 & d_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_1' \end{bmatrix} = \begin{bmatrix} 1 & d_1 + d_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_1' \end{bmatrix}$$

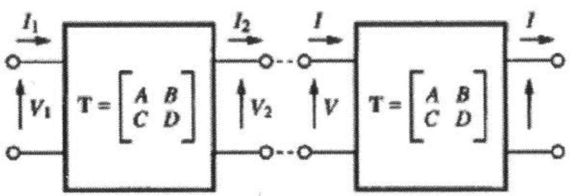
can be checked by inserting $d_1 + d_2$ for d for the cascade

note: the similarity with electrical networks (see below Fig. 2.3)

note: for a bilateral network $\det[T] = AD - BC = 1$

bilateral network $\equiv$ a circuit where $|I|$ remains the same, when voltage is reversed

This is also true for optical rays provided n (index of refraction) is the same in the exit as in the input plane



$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_1 \\ I_1 \end{bmatrix}$$

FIGURE 2.3. Correspondence with electrical network theory.

Some common ray matrices

thin lens: position $r_2 = r_1$

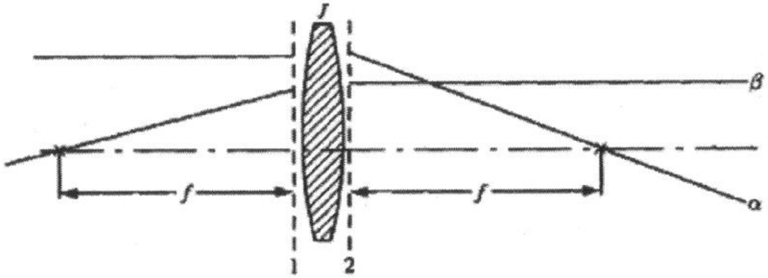


FIGURE 2.4. Paper experiment with a "thin" lens.

consider special case α: $r_1' = 0$ 'input slope'

$$r_2' = -\frac{r_1}{f}$$

$$r_{2\alpha}' = C r_{1\alpha} + D r_{1\alpha}' = -\frac{r_{1\alpha}}{f} = C r_{1\alpha} + D \cdot 0$$

$$C = -\frac{1}{f}$$

consider special case β: $r_1' = \frac{r_1}{f}$ $r_2' = 0$

$$r_{2\beta}' = 0 = -\frac{1}{f} r_{1\beta} + D r_{1\beta}' \rightarrow r_{1\beta}' = \frac{r_{1\beta}}{f}$$

$$D = 1$$

thin lens

$$T = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$$

note: $\mathrm{Det}\, T = 1$

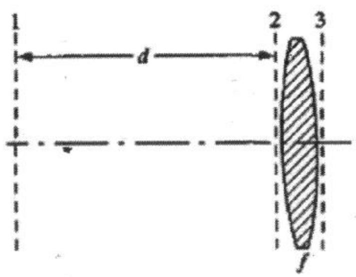


FIGURE 2.5. Combination of a lens plus free space.

combines free space and thin lens in cascade

$$T = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & d \\ -\frac{1}{f} & 1 - \frac{d}{f} \end{bmatrix}$$

note: rays in reverse order as the ray goes through mirror:

to an observer riding with the ray, the effect of a spherical mirror of radius R is to direct the ray twowards the axis just like a ~~thin~~ lens

$f = \frac{R}{2}$

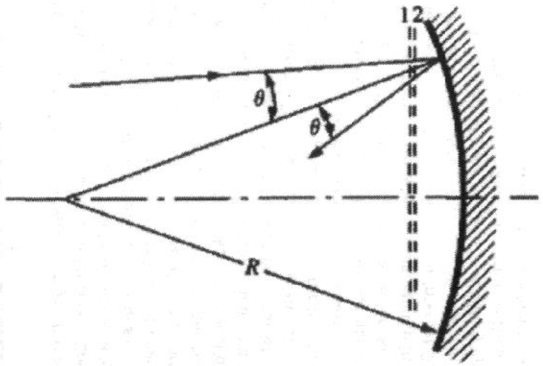


FIGURE 2.6. Mirror. Note that the entrance (1) and exit (2) planes are on the same side of the mirror.

$$T = \begin{bmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{bmatrix}$$

spherical mirror

Optical cavity

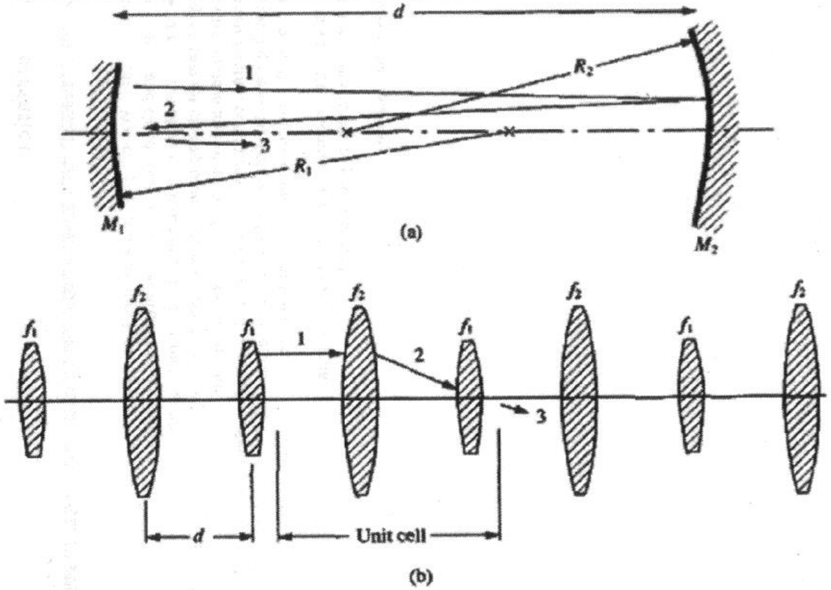


FIGURE 2.7. (a) Optical cavity showing a ray bouncing back and forth between the mirrors; (b) lens-waveguide equivalent to the mirror system shown in (a).

$$\mathbf{T} = \begin{bmatrix} 1 & d \\ -\frac{1}{f_1} & \left(1 - \frac{d}{f_1}\right) \end{bmatrix} \begin{bmatrix} 1 & d \\ -\frac{1}{f_2} & 1 - \frac{d}{f_2} \end{bmatrix}$$

$$\mathbf{T} = \begin{bmatrix} 1 - \frac{d}{f_2} & d + d\left(1 - \frac{d}{f_2}\right) \\ -\frac{1}{f_1} - \frac{1}{f_2}\left(1 - \frac{d}{f_1}\right) & \left(1 - \frac{d}{f_1}\right)\left(1 - \frac{d}{f_2}\right) - \frac{d}{f_1} \end{bmatrix}$$

ray bounces back and forth between the mirrors

stable: ray stays close to the axis after many bounces

unstable: ray walks of the axis

conditionally stable: only after careful alignment stays close to axis

Let us find a second-order difference equation for the ray as it passes the various succeeding unit cells

$$r_{s+1} = A r_s + B r_s' \quad \text{or} \quad r_s' = \frac{1}{B}(r_{s+1} - A r_s)$$

$$C r_s + D r_s' = r_{s+1}' = \frac{1}{B}(r_{s+2} - A r_{s+1})$$

substituting r_s'

$$\frac{1}{B}(r_{s+2} - A r_{s+1}) = C r_s + \frac{D}{B}(r_{s+1} - A r_s)$$

combining terms and remembering $AD - BC = 1$

$$\boxed{r_{s+2} - 2\left(\frac{A+D}{2}\right) r_{s+1} + r_s = 0}$$

solve the difference equation:

$$\boxed{r_{s+2} - 2\left(\frac{A+D}{2}\right) r_{s+1} + r_s = 0}$$

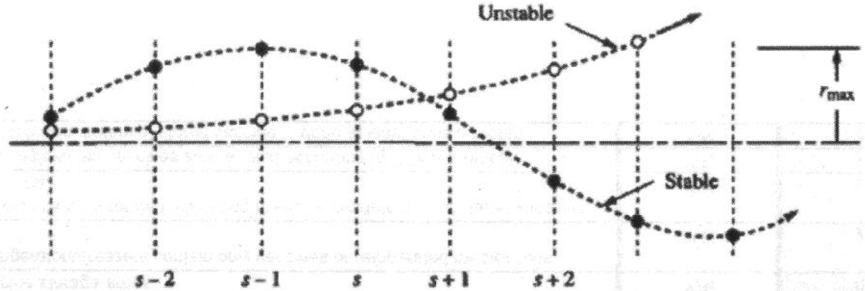


FIGURE 2.8. Example of the ray's position at the various planes of the lens waveguide.

Are there solutions in which $|r| \le r_0$?

Answer: yes → stable; no → unstable } see Fig. 2.8

trial solution: $r_s = r_0 \left(e^{j\Theta s}\right)$

r must be real → we will get two solutions to the second order difference equation, which when combined yield a real number for r

$$\underbrace{r_0}_{\text{is not } = 0} e^{js\Theta}\underbrace{\left[e^{j2\Theta} - 2\,\frac{A+D}{2}\,e^{j\Theta} + 1\right]}_{=0} = 0$$

This is a quadratic equation in $e^{j\Theta}$ with two solutions

$$\boxed{e^{j\Theta} = \frac{A+D}{2} \pm j\left[-\left(\frac{A+D}{2}\right)^2 + 1\right]^{1/2}}$$

if all quantities are real the solutions are complex conjugates

$$r_s = r_0 e^{js\Theta} + r_0^* e^{-js\Theta}$$

$$\boxed{r_s = r_{max} \sin(s\Theta + \alpha)}$$

Stability diagram

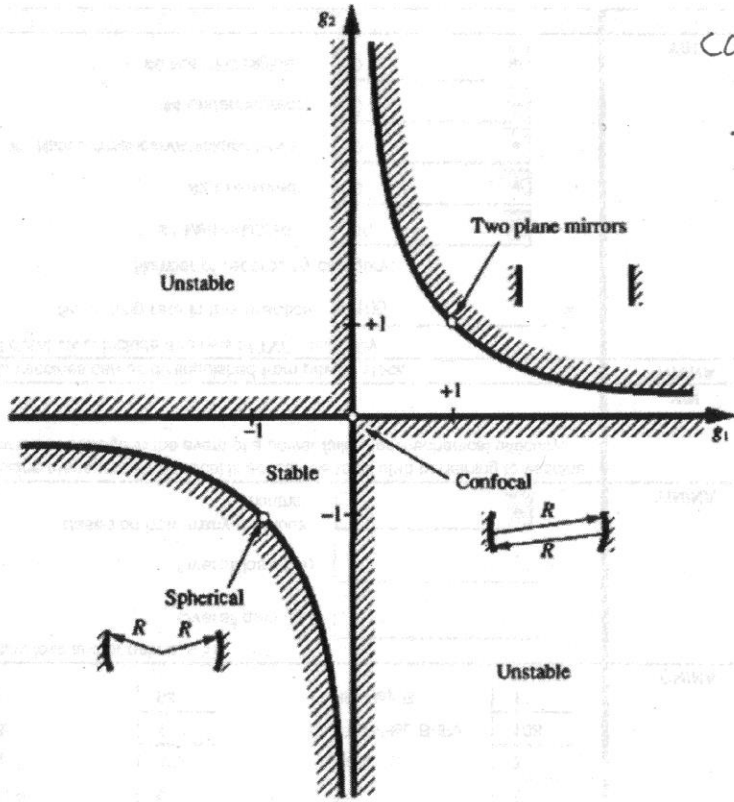


FIGURE 2.9. Stability diagram for the cavity of Fig. 2.7.

condition that expression is real

$$-1 \le \frac{A+D}{2} \le 1 \quad \Big| \text{ add } +1, \text{ divide by } 2$$

$$\boxed{0 \le \frac{A+D+2}{4} \le 1}$$

consider two-spherical mirror cavity:

$$\frac{A+D+2}{4} = \frac{1}{4}\left[1 - \frac{d}{f_2} - \frac{d}{f_1} + \left(1 - \frac{d}{f_2}\right)\left(1 - \frac{d}{f_1}\right) + 2\right]$$

$$= 1 - \frac{d}{2f_1} - \frac{d}{2f_2} + \frac{d^2}{4f_1 f_2} = \left(1 - \frac{d}{2f_1}\right)\left(1 - \frac{d}{2f_2}\right)$$

since $2f_1 = R_1$ and $2f_2 = R_2$
we get for this special case a very simple expression for the stability of the cavity

$$\boxed{0 \le g_1 g_2 \le 1} \quad \text{with } g_{1,2} = 1 - \frac{d}{R_{1,2}}$$

This relation is graphed

A note on an *unstable* cavity. Not just stable but also unstable cavities are useful.

$$\boxed{(A+D)^2 > 1 \quad \text{unstable}}$$

Consider a high gain medium where the single pass gain is 5
Consider only 10 passes before the ray gets out of the cavity

exit power: $5^{10} = 9.76 \times 10^6 \approx \underset{\text{total gain}}{10^7}$

$$\boxed{\text{input } 1\,\text{mW} \times \underset{\text{gain}}{10^7} = 10\,\text{kW}}$$

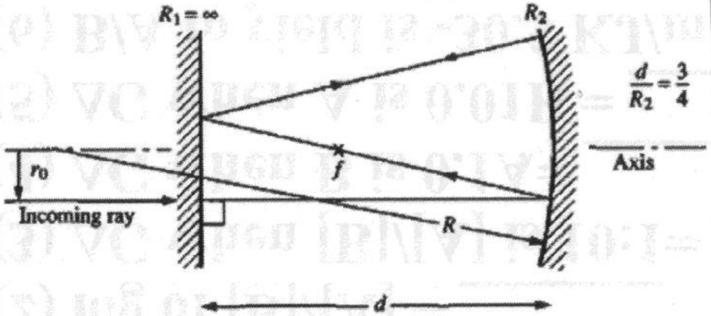


FIGURE 2.10. Ray tracing in a stable cavity.

$$\mathbf{T}_1 = \begin{bmatrix} 1 - \frac{d}{f} & d + d\left(1 - \frac{d}{f}\right) \\ -\frac{1}{f} & 1 - \frac{d}{f} \end{bmatrix}$$

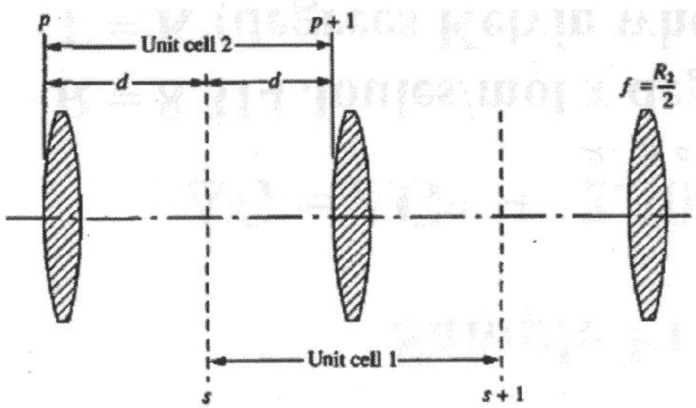


FIGURE 2.11. Lens-waveguide equivalent to the cavity of Fig. 2.10.

Example of ray tracing in a stable cavity

This cavity shown (consisting of flat and spherical mirror with $\frac{d}{R_2} = \frac{3}{4}$ is stable.

$$g_1 = 1 - d/R_1 = 1 \qquad R_1 = \infty$$

$$g_2 = 1 - d/R_2 = \frac{1}{4}$$

$$g_1 \cdot g_2 = \frac{1}{4} < 1$$

lens-waveguide equivalent

$$\boxed{e^{j\Theta} = \frac{A+D}{2} + j\left[1 - \left(\frac{A+D}{2}\right)^2\right]^{1/2} = \cos\Theta + j\sin\Theta}$$

or $\cos\Theta = \frac{A+D}{2} = 1 - \frac{d}{f}$

since $f = \frac{R}{2}$ $\quad \cos\Theta = 1 - \frac{3}{2} = -\frac{1}{2} = \frac{2\pi}{3}$

$\Theta = 120°$

Astigmatism:

optical path through Brewster angle window

$$\boxed{d_x = t\frac{(n^2+1)^{1/2}}{n^4} \qquad d_y = t\frac{\left(n^2+1\right)^{1/2}}{n^2}}$$

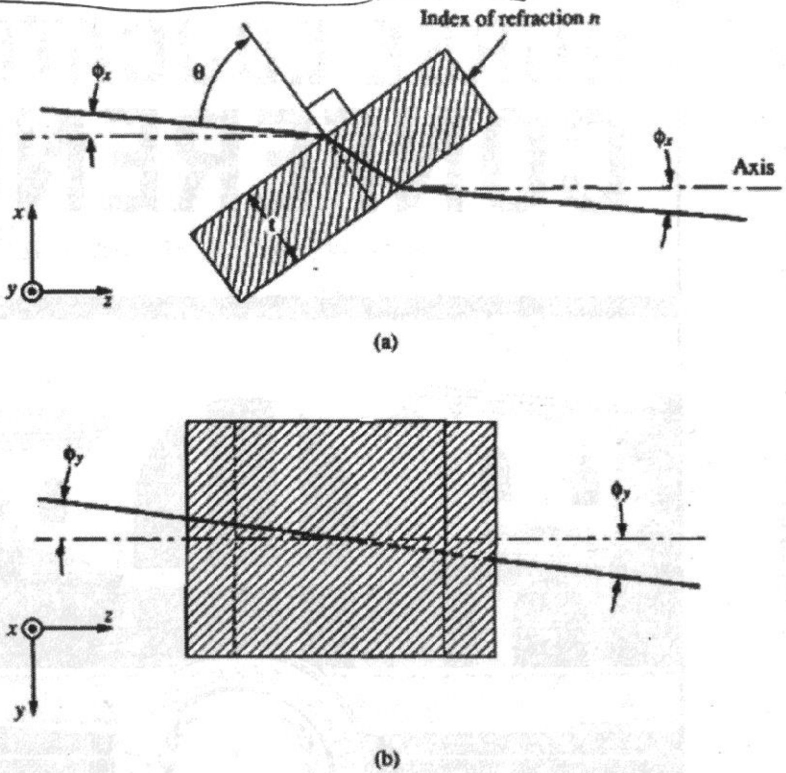


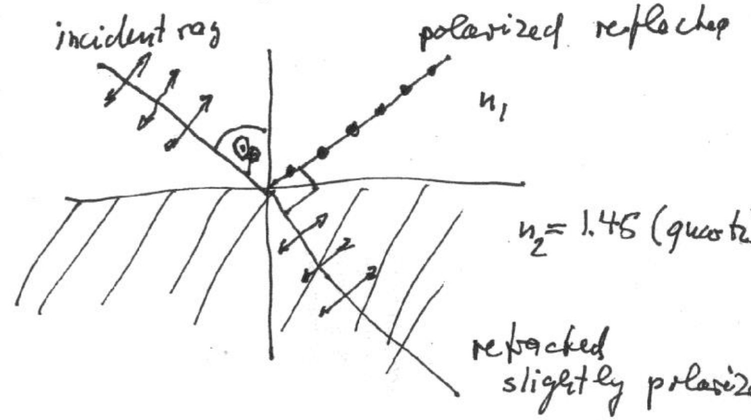
FIGURE 2.12. Astigmatism of a window. (a) Side view. (b) Top view.

detailed description by Fresnel equation, here we give a geometry argument for Θ_B

$$\boxed{\text{Brewsters angle} \quad \Theta_B = tg^{-1}\, n_2/n_1}$$

$n_1(\text{air}) \approx 1$

for quartz $n_2 = 1.458 \Rightarrow 55.5^0$



$\Theta_1 + \Theta_2 = 90$ no radiation emitted $\perp$ to dipole

Θ_1 = incident angle

Θ_2 = refractive angle

$n_1 \sin\Theta_1 = n_2 \sin\Theta_2$

the angle $\Theta_1 = \Theta_B$ where no light is reflected

$$n_1 \sin\Theta_B = n_2 (90 - \Theta_B) = n_2 \cos\Theta_B$$

$$tg\,\Theta_B = \frac{\sin\Theta_B}{\cos\Theta_B} = \frac{n_2}{n_1}$$

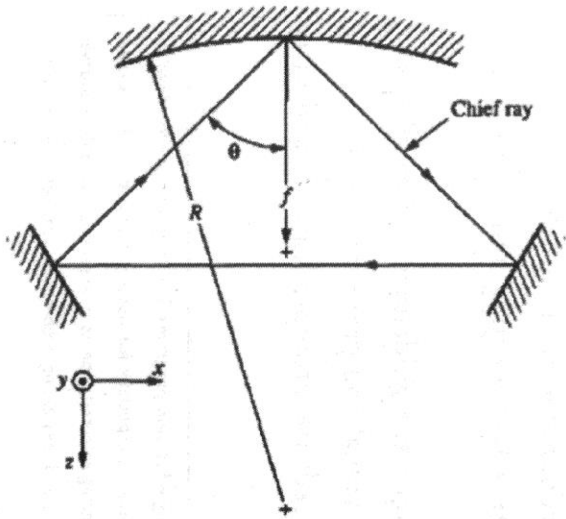


FIGURE 2.13. Astigmatic laser cavity.

The mirror focusses parallel rays in the two planes in different locations leading to different effective focal lengths in the xy and xz planes

Thus for rays that are paraxial to the "chief ray we use an effective focal length

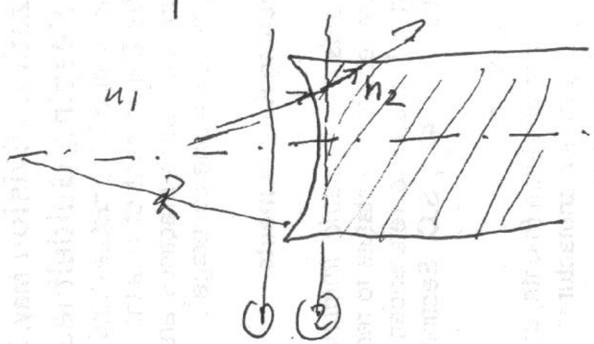
$$f_x = f\cos\theta$$
$$f_y = \frac{f}{\cos\theta}$$

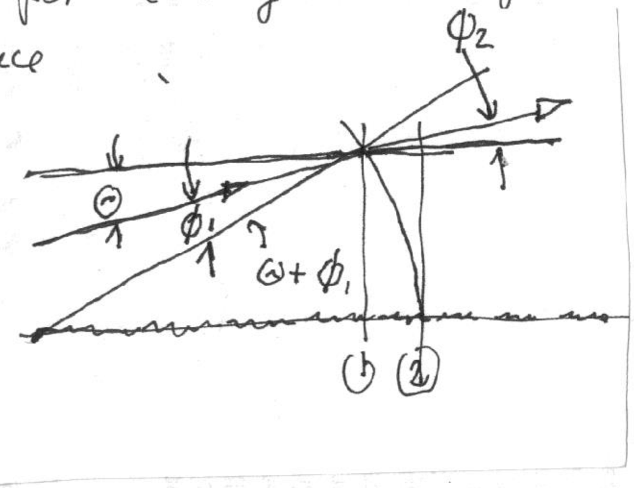
$$d_x = t\frac{(n^2+1)^{1/2}}{n^4}$$

$$f_x = f\cos\theta$$
$$f_y = \frac{f}{\cos\theta}$$

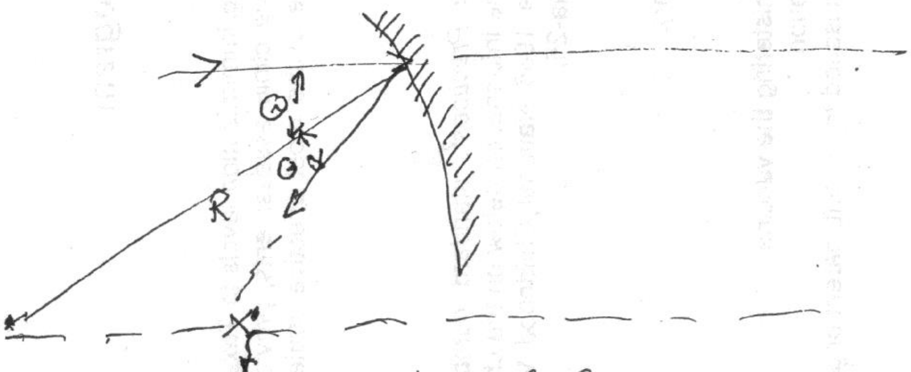
Astigmatism leads to elliptical beams in ring laser cavities and is critical in dye laser cavities. There are folded cavities where astigmatism is compensated by a proper choice of the folding angle

2.1 Derive the ray matrix for a ray entering a spherical dielectric interface





$$T=\begin{bmatrix}1 & 0\\ \left(1-\frac{n_1}{n_2}\right)\frac{1}{R} & \frac{n_2}{n_1}\end{bmatrix}$$



spherical lens

$$n_1 \sin\phi_1 = n_2 \sin\phi_2$$

$$n_1\phi_1 = n_2\phi_2 \quad \text{(paraxial)}$$

$$r_2' = \Theta+\phi_1-\phi_2 = \Theta+\phi_1\left(1-\frac{n_1}{n_2}\right)$$

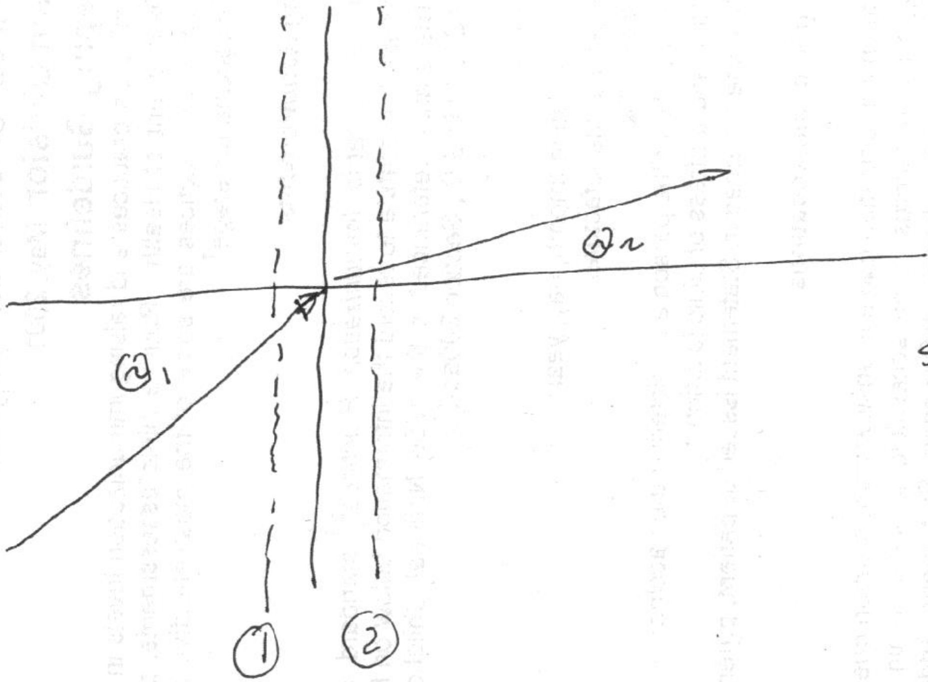
$$\frac{r_1}{R} = \sin(\Theta+\phi_1) = \Theta+\phi$$

$$\phi_1 = \frac{r_1}{R} - r_1'$$

$$r_2' = r_1' + \left(1-\frac{n_1}{n_2}\right)\left(\frac{r_1}{R}-r_1'\right) = \underbrace{\left\{\left[1-\frac{n_1}{n_2}\right]\frac{1}{R}\right\}}_{B} r_1 + \underbrace{\frac{n_1}{n_2}}_{D} r_1'$$

$$T=\begin{bmatrix}1 & 0\\ -\frac{2}{R} & 1\end{bmatrix}$$

2.2. Derive the ray matrix for the plane dielectric interface



the distance between (1) and (2) = 0

so $r_2 = r_1$ and $A = 1$ $B = 0$

Snell's law

$$n_1 \sin \theta_1 = n_2 \sin \theta_2$$

for paraxial ray $\sin \theta \approx \theta$

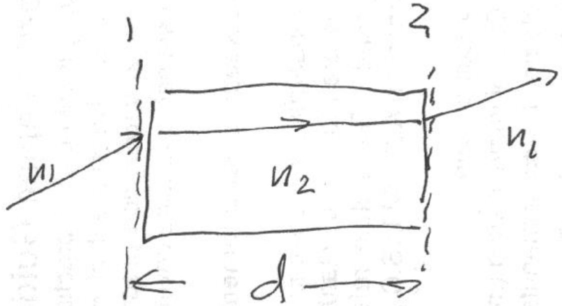
$$\theta_2 = r_2' = \frac{n_1}{n_2} \theta_1 = \frac{n_1}{n_2} r_1'$$

and $C = 0, D = \frac{n_1}{n_2}$

$$\begin{bmatrix} r_{out} \\ r'_{out} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} r_{in} \\ r'_{in} \end{bmatrix}$$

$$\boxed{T = \begin{bmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{bmatrix}}$$

2.3 Find ray matrix for the plane dielectric slab of thickness d



$$\begin{bmatrix} r_2 \\ r_2' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{n_2}{n_1} \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{bmatrix} \begin{bmatrix} r_1 \\ r_1' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{n_2}{n_1} \end{bmatrix} \begin{bmatrix} 1 & \frac{n_1}{n_2} d \\ 0 & \frac{n_1}{n_2} \end{bmatrix} \begin{bmatrix} r_1 \\ r_1' \end{bmatrix} = \begin{bmatrix} 1 & \frac{n_1}{n_2} d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_1' \end{bmatrix}$$

Home work:

2.4 Combine the results of problems 2.1 and 2.2 to derive the ray matrix for the negative lens ($R \gg d$)