Resonant optical resonators

4. Resonant optical resonators

Resonant optical cavities

(a) Why does a laser always oscillate at a cavity resonance?

(b) What is the photon lifetime?

(c) Why does the cavity filter the spontaneous emission of the atoms, so that stimulated emission takes place?

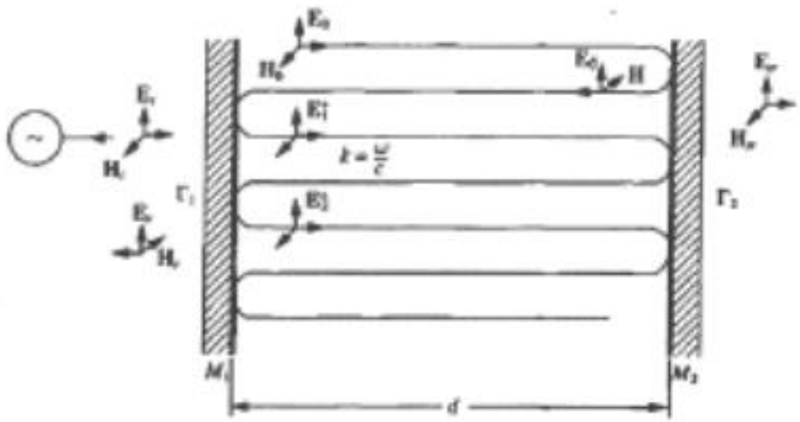


FIGURE 6.1. Optical cavity.

Consider all waves: incident on the cavity from left, inside the cavity, transmitted through the cavity } to be uniform plane waves of limited spatial transverse extend

$$\boxed{\text{Condition of resonce: Round Trip Phase shift } RTPS = 2kd = q\,2\pi}$$

① find the resonant wave lengths

$$k \cdot 2d = \frac{\omega n\, 2d}{c} = \frac{2\pi}{\lambda} 2d = q\,2\pi \qquad \lambda = \frac{2d}{q} \qquad q = \text{very large number}$$

$d = \frac{q \cdot \lambda}{2}$ There is an integral of half wavelengths between the mirrors

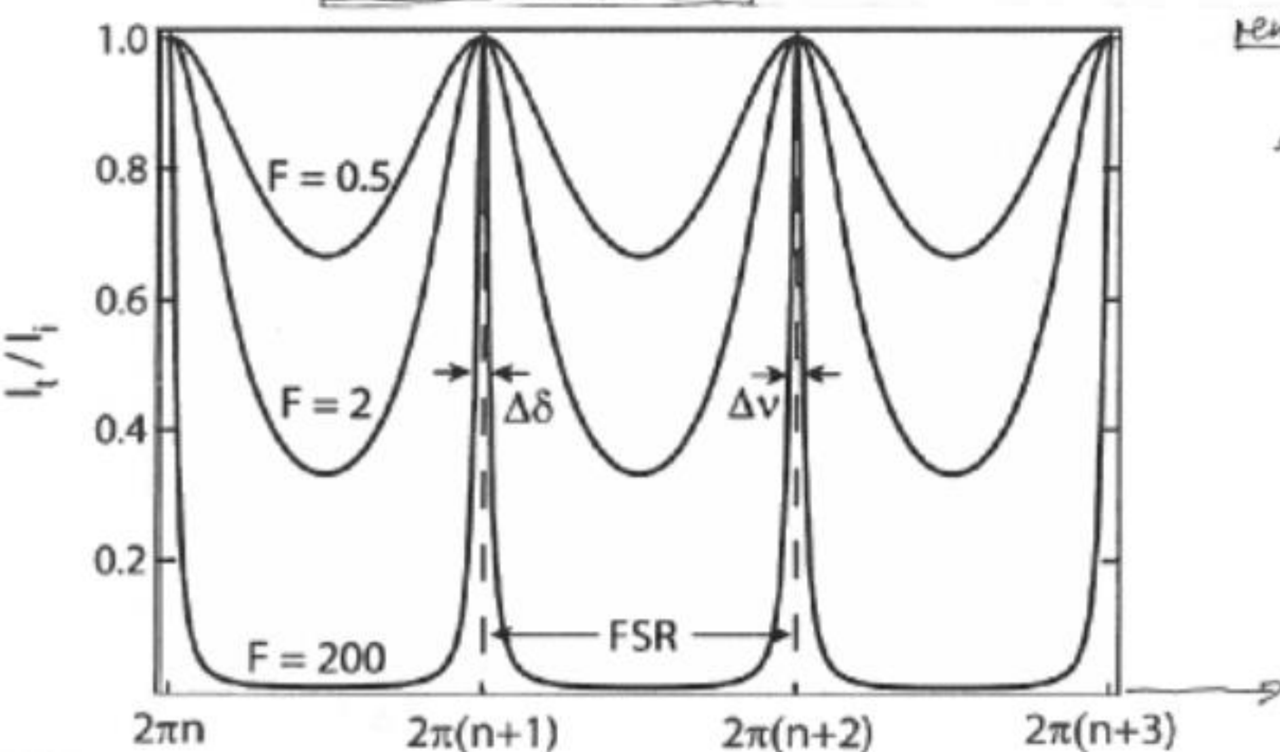
2. find frequencies for different q

$$k\cdot 2d = \omega\cdot\frac{2nd}{c} = 2\pi\nu\frac{2nd}{c} = q\,2\pi$$

$$\nu = q\,\frac{c}{2nd}$$

The separation between those neighboring frequencies

$$\boxed{\nu_{q+1} - \nu_q = \frac{c}{2nd}} = \text{free spectral range}$$



remember:

$ex\{-j(kz)\}$ = phase

$kz = \phi = \Theta$

$\phi_{RTP} = 2kd = q\,2\pi$

$q = n+1, n+2 \ldots$
large numbers

Sharpness of the resonance: Q-factor, F = finesse, τ_{photon} = photon life time

$$E_T^+ = \Sigma E_N^+ = \underbrace{\mathbf{E}_0[1}_{\text{incident}} + \underbrace{\Gamma_1\Gamma_2 e^{-jk\cdot 2d}}_{\text{one}} + \underbrace{(\Gamma_1\Gamma_2 e^{-jk\cdot 2d})^2}_{\text{two}} + \cdots \quad \text{round trips}$$

total field propagating to the right

$$= \mathbf{E}_0\left[\frac{1}{1-\Gamma_1\Gamma_2 e^{-j2\theta}}\right] \tag{6.3.1}$$

$\left[\mathcal{E}_0 \equiv E_0 e^{-j\omega t}\right.$ with time dependent part$\left.\right]$

$$E_T^- = \Gamma_2 e^{-j2\theta}E_T^+ = \mathbf{E}_0\left[\frac{\Gamma_2 e^{-j2\theta}}{1-\Gamma_1\Gamma_2 e^{-j2\theta}}\right] \tag{6.3.2}$$

total field propagating to the left

$$I^+(z=0^+) = \frac{|\mathbf{E}_0|^2}{2\eta}\left\{\frac{1}{1-\Gamma_1\Gamma_2 e^{-j2\theta}-(\Gamma_1\Gamma_2 e^{-j2\theta})^* + |\Gamma_1\Gamma_2|^2}\right\}$$

$$= I_0\frac{1}{1-2|\Gamma_1\Gamma_2|\cos 2\theta + |\Gamma_1\Gamma_2|^2}$$

$$I^+(z=0^+) = I_0\frac{1}{1-2|\Gamma_1\Gamma_2|[1-2\sin^2\theta] + |\Gamma_1\Gamma_2|^2}$$

$$I^+(z=0^+) = \frac{E_0^2}{2\eta}\left\{\frac{1}{(1-\sqrt{R_1R_2})^2 + 4\sqrt{R_1R_2}\sin^2\theta}\right\} \tag{6.3.3}$$

$$I_t = \left\{\frac{E_0^2}{2\eta} = T_1 I_{\text{inc}} = (1-R_1)I_{\text{inc}}\right\}\left\{\frac{T_2 = (1-R_2)}{(1-\sqrt{R_1R_2})^2 + 4\sqrt{R_1R_2}\sin^2\theta}\right\}$$

or

$$\boxed{T(\theta) = \frac{I_{\text{trans}}}{I_{\text{inc}}} = \left\{\frac{(1-R_1)(1-R_2)}{(1-\sqrt{R_1R_2})^2 + 4\sqrt{R_1R_2}\sin^2\theta}\right\}} \tag{6.3.4}$$

The net transmission is a maximum when the denominator is a minimum

Example 1 : $R_1 = R_2$ and $\sqrt{R_1 R_2} = R$ at resonance $\sin^2\theta = 0$

$$T(q\pi) = \frac{(1-R)^2}{(1-R)^2} = 1$$

This is amazing! Assume $R = 99\%$ one would think transmission through the 2 mirrors is 10^{-4}. This is wrong since the waves inside the cavity are coherent and the fields add up.

Also transmission through M_2 is $1-R$ so I^+ falling on M_2 from the right must be $\frac{1}{1-R}$ ~~so that~~ of the incident value I_{inc} so that the product would be I_{inc}. That means $I^+ = 100$ times I_{inc}

incident on cavity

$\boxed{\text{The running waves inside the cavity are much larger than those on the outside}}$

This manifests the energy storage capability of the F-P cavity!

In antiresonance $T[\theta = (q+\frac{1}{2})\pi] = 2.53\times10^{-5}$ for $R = 99\%$

resonance condition

unit on graph below on horizontal axis

$\frac{\Theta}{\pi} - q = 2q - q = q$

vertical axis is the transmission coefficient: which is a measure of the relative fields, energy, intensities of running fields inside the cavity

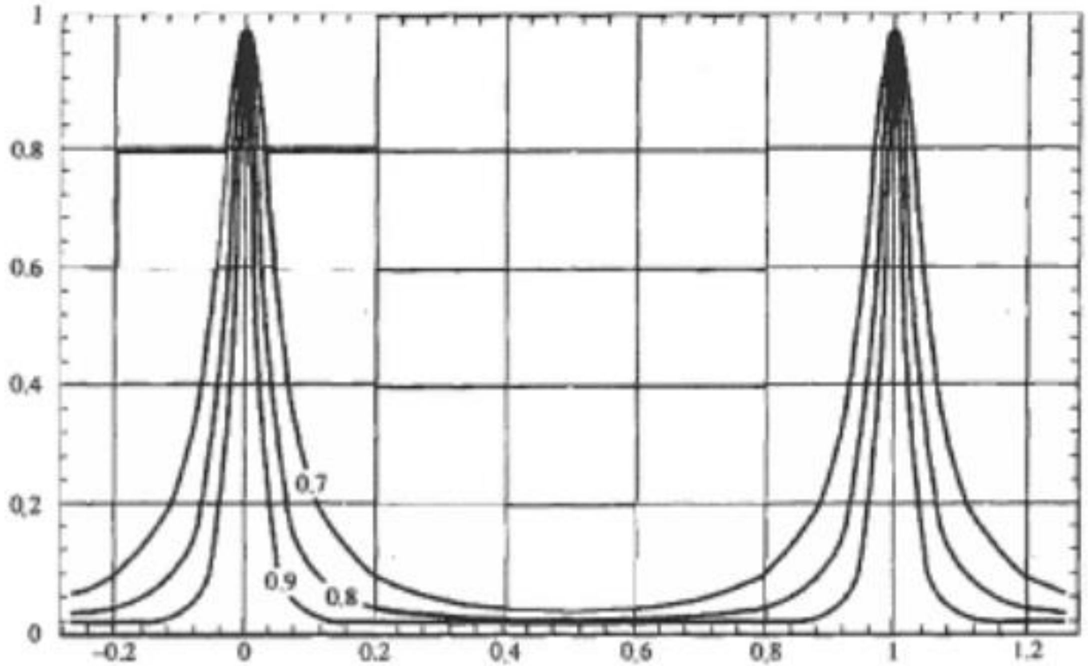


FIGURE 6.3. The transmission through a Fabry-Perot cavity as a function of the electrical length measured in units of $\theta/\pi - q$. The three curves were plotted for $R_1 = R_2 = 0.9$, 0.8, and 0.7.

one can work out ν_+ and ν_- = frequencies where T has fallen to 50% of maximum

without proof:

$$\Delta\nu_{1/2} = \nu_+ - \nu_- = \frac{c}{2nd}\left\{\frac{1-(R_1R_2)^{1/2}}{\pi(R_1R_2)^{1/4}}\right.$$

quality factor

$$\boxed{Q = \frac{q(c/2nd)}{\Delta\nu} = \frac{2\pi nd}{\lambda_0}\frac{(R_1R_2)^{1/4}}{1-(R_1R_2)^{1/2}} \qquad q = \frac{nd}{\lambda_0/2}}$$

Q is very large $\approx 10^8$

Introduce another measure of sharpness F = finesse

$$\boxed{F = \frac{FSR}{\text{full width at half max}} = \frac{c/2nd}{\Delta\nu_{1/2}} = \frac{\pi(R_1R_2)^{1/4}}{1-(R_1R_2)^{1/2}}}$$

Photon life time

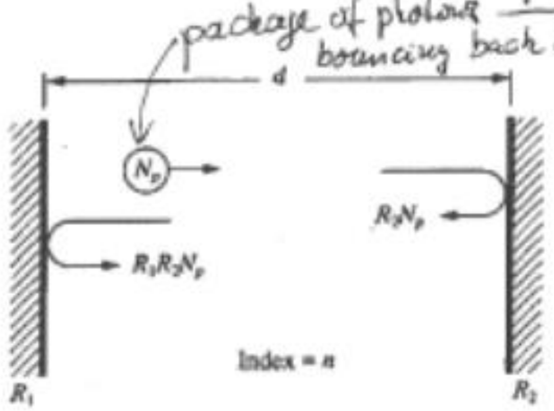


FIGURE 6.4. Decay of photons in a cavity.

τ_{phot} = describes build up or decay of the energy in a cavity

use rate equation = diff. equation for $f(t)$

energy in cavity : $h\nu N_p$, $S = R_1 R_2$ = survival factor

# of photons lost in one round trip $= (1-S)N_p$

$$\frac{dN_p}{dt} = \frac{N_p(t+\tau_{RT}) - N_p(t)}{\tau_{RT}} = -\frac{(1-S)N_p}{\tau_{RT}}$$

$$\boxed{\frac{dN_p}{dt} = -\frac{N_p}{\tau_p}} \qquad \boxed{\tau_p = \frac{\tau_{RTP}}{1-S} \text{ photon life time}} \qquad N_p(t) = N_p(0)\exp\left[-t/t_p\right]$$

$\tau_{RTP} = 2nd/c$ $\quad S = R_1 R_2 \exp[-\alpha/2d]$ if there is absorption

$$Q = \frac{2\pi\,(\text{energy stored in the system at resonance} = W)}{\text{energy lost in a cycle of oscillation}}$$

T = period $= \frac{1}{\nu}$

$$Q = \frac{2\pi W}{T\langle P\rangle} = \omega_0 \frac{W}{\langle P\rangle} = \omega_0 \frac{W}{-\frac{dW}{dt}} \quad \text{or} \quad \frac{dW}{dt} = -\frac{\omega_0}{Q}\cdot W$$

$$\frac{dW}{dt} = -\frac{\omega_0}{Q}W \Rightarrow \frac{d}{dt}(h\nu N_p) = \frac{\omega_0}{Q}[h\nu N_p] = -\frac{h\nu N_p}{\tau_p} \qquad \boxed{\tau_p = \frac{Q}{\omega_0}}$$

Summary of results from photon life time

$$\tau_p = \frac{\tau_{RT}}{1-S} = \frac{2nd/c}{1-R_1R_2} \qquad \text{by (6.4.2b)}$$

$$\therefore \Delta\nu_{1/2} = \frac{1}{2\pi\tau_p} = \frac{c(1-R_1R_2)}{(2nd)\cdot 2\pi} \qquad \text{by (6.4.5)}$$

$$Q = \frac{\nu_0}{\Delta\nu_{1/2}} = 2\pi\nu_0\left(\frac{2nd}{c}\right)\frac{1}{1-R_1R_2} = \frac{4\pi nd}{\lambda_0}\left(\frac{1}{1-R_1R_2}\right) \qquad \text{by (6.3.5)}$$

$$F = \frac{FSR}{\Delta\nu_{1/2}} = \frac{c/2nd}{\Delta\nu_{1/2}} = \frac{2\pi}{1-R_1R_2} \qquad \text{by (6.3.8)}$$

These formulas are a little different from the one the E field travelling back and forth inside the cavity, but for large values of Q or F they are practically the same

Resonance of the Hermite – Gaussian Modes

RTPS $= q\,2\pi$ is still the resonance condition
The phase shift is not $\omega n l/c$ but more complicated

For TEM_{mp} : $\phi(d) - \phi(0) = \underline{kd - \left((1+m+p)\tan^{-1}\frac{d}{z_0}\right)}$

$$\boxed{kd - (1+m+p)\tan^{-1}\left(\frac{d}{z_0}\right) = q\pi}$$

FIGURE 6.5. Optical cavity for the Hermite-Gaussian beam modes.

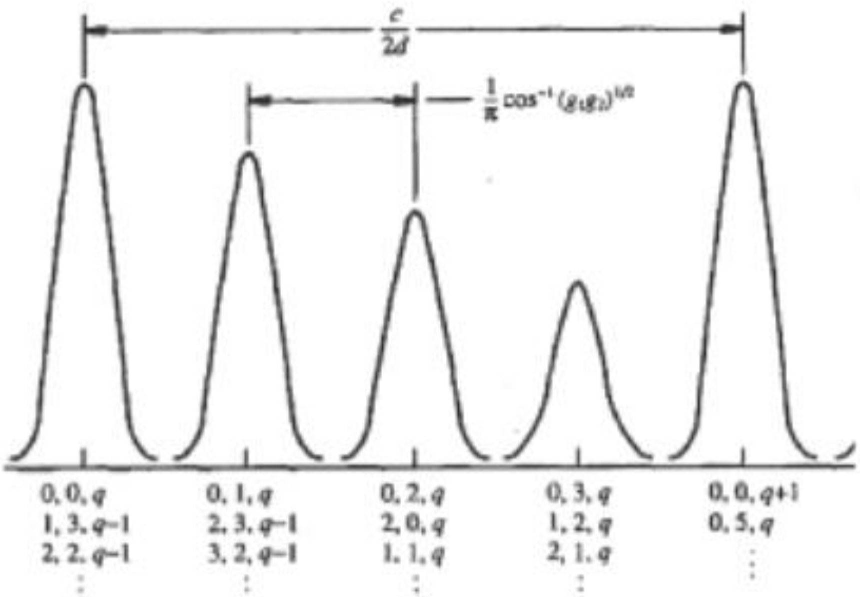


FIGURE 6.6. Frequency degeneracy in an optical cavity.

If we recall (5.2.2), which relates z_0 to the radius of curvature, R_2,

$$z_0 = (dR_2)^{1/2}\left(1 - \frac{d}{R_2}\right)^{1/2} \tag{5.2.2}$$

and perform some painful, yet trivial manipulations on (6.5.1), we obtain a more easily interpreted formula for the resonant frequencies of the $\text{TEM}_{m,p,q}$ modes.

$$\nu_{m,p,q} = \frac{c}{2nd}\left\{q + \frac{1+m+p}{\pi}\tan^{-1}\left[\frac{(d/R_2)^{1/2}}{(1-d/R_2)^{1/2}}\right]\right\} \tag{6.5.2}$$

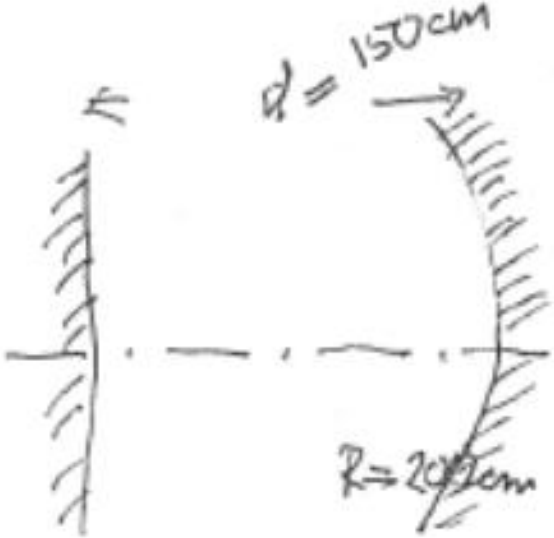
or

$$\nu_{m,p,q} = \frac{c}{2nd}\left[q + \frac{1+m+p}{\pi}\cos^{-1}\left(1 - \frac{d}{R_2}\right)^{1/2}\right] \tag{6.5.3}$$

If M_1 has a finite radius of curvature R_1, the arithmetic is most painful, but the answer is quite similar to (6.5.3):

$$\nu_{m,p,q} = \frac{c}{2nd}\left[q + \frac{1+m+p}{\pi}\cos^{-1}(g_1g_2)^{1/2}\right] \tag{6.5.4}$$

problems



6.1. The following questions refer to the optical cavity shown in Fig. 6.5 with $d = (3/4)R_2$, $\Gamma_1^2 = 0.99$, and $\Gamma_2^2 = 0.97$.

(a) Find an expression for the resonant frequencies of the $TEM_{0,0}$ modes of the cavity.

(b) If the radius of curvature is 2.0 m and the wavelength region of interest is 5000 Å, compute the following quantities:

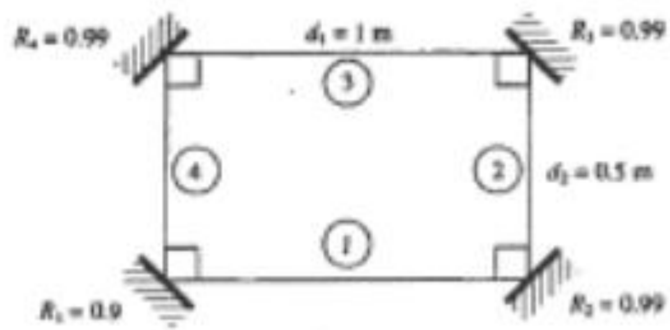
(1) Free spectral range in MHz and in Å units

(2) Cavity Q

(3) Photon lifetime in nsec

(4) Finesse

Problems 6.2 through 6.4 refer to the optical cavity shown in the accompanying diagram.



6.2. If the optical paths 1 through 4 are lossless, what is the photon lifetime of this cavity? (Ans.: 78.9 ns.)

6.3. What is the cavity Q (assume that the wavelength region of interest is 5000 Å)? (Ans.: 2.97×10^8.)

solution 6.1

(a) $\nu_{00q} = \frac{c}{2d}\left\{q+\frac{1}{\pi}\left[\cos^{-1}(1-\frac{3}{4})^{1/2} = \frac{\pi}{6}\right] = \frac{c}{2d}\left\{q+\frac{1}{6}\right\}\right.$

(b) $FSR = \frac{c}{2d} = 100\,MHz$ $\quad \nu = 6\times10^{14}\,Hz$

$\frac{d\nu}{\nu} = \frac{d\lambda}{\lambda} \quad d\lambda = \lambda\frac{d\nu}{\nu} = 8.3\times10^{-4}\,\mathring{A}$

$\tau_p = \frac{2d/c}{1-R_1R_2} = 252\,ns$ $\quad Q = \frac{\nu_0}{\Delta\nu_{1/2}} = 9.5\times10^{8}$

$\Delta\nu_{1/2} = \frac{1}{2\pi\tau_p} = 632\,kHz$ $\quad F = \frac{c/2d}{\Delta\nu_{1/2}} = \frac{100\times10^{6}}{0.632\times10^{3}} = 158$

solution: 6.2 $\quad \tau_p = \frac{\text{roundtrip time}}{\text{fraction lost}} = \frac{2(d_1+d_2)/c}{1-R_1R_2R_3R_4} = 78.9\,ns$

solution 6.3 $\quad \lambda_0 = 5000\,A \quad \nu_0 = 6\times10^{14}\,Hz = 600\,THz \quad \Delta\nu_{1/2} = \frac{1}{2\pi\tau_p} = 2.02\,MHz$

$Q = \frac{\nu_0}{\Delta\nu_{1/2}} = 3\times10^{8}$

6.1. The following questions refer to the optical cavity shown in Fig. 6.5 with $d = (3/4)R_2$, $\Gamma_1^2 = 0.99$, and $\Gamma_2^2 = 0.97$.

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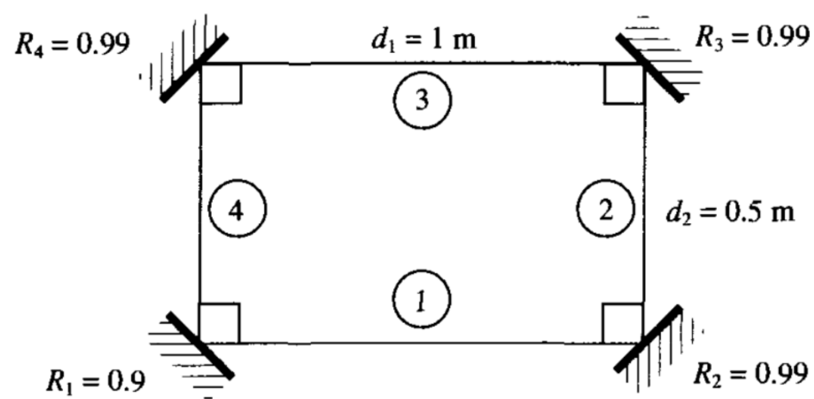

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6.4. **(a)** Suppose that path 1 has a transmission coefficient of 0.85 rather than 1 as in Problem 6.2. What is the new photon lifetime? (Ans.: 38.8 nsec.)

(b) Suppose that path 1 had a power gain of 1.1. What is the new photon lifetime?